ANR HighAGT 2022

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Goal

- We want resolutions of algebras (in a general sense).
- Why?
  - Compute homotopical invariants: derived tensor products, derived mapping spaces, André-Quillen homology…
  - Define homotopy algebras over operads.
- Tool of choice: Koszul duality [Priddy…]

Quadratic algebras

Koszul duals

- Starting data: $A = T(E)/(R)$, $E \subset R \otimes R$.
- Koszul dual coalgebra: $A^{¡} = T^c\left(\Sigma E, \Sigma^2 R\right)$.
- Often easier to write down Koszul dual algebra: $A^! = T(E^*)/\left(R^\perp\right)$.
- Examples:
  - $A = T(E)$, $R = 0 \Longrightarrow A^! = E^*$ with trivial multiplication.
  - $A = S(E) = T(E)/\left(xy - yx\right) \Longrightarrow A^! = \Lambda(E^*) = T(E^*)/\left(\alpha\beta + \beta\alpha\right)$.
- Koszul complex: $K_A = (A \otimes A^{¡}, d(\Sigma e) = e)$.
- A is Koszul if $\widetilde{H}\left(K_A\right) = 0$.

Resolutions

- Bar/cobar: Ω: $\{\text{coaug. coalg.}\} \leftrightarrows \{\text{aug. alg.}\}: B$, where
  $\Omega C = \left(T\left(\Sigma \bar{C}\right), d\right), \qquad BA = \left(T^c\left(\Sigma \bar{A}\right), d\right)$
- Canonical resolution $\Omega BA \to A$… but it is very big!
- A quadratic $\Longrightarrow$ canonical morphism $\Omega A^{¡} \to A$
- Theorem [Priddy, Positselski]: A is Koszul iff $\Omega A^{¡} \to A$ is a quasi-isomorphism.
- Get much smaller resolutions:
  - $A = T(E) \Rightarrow \Omega A^{¡} = A = T(E)$ vs. $\Omega BA = TT^cT(E)$.
  - $A = S(E) \Rightarrow \Omega A^{¡} = T\Lambda^c(E)$ vs. $\Omega BA = TT^cS(E)$.

Unital algebras

Koszul duals

- Quadratic-linear-constant algebra: $A = T_+(E)/(R)$, $R \subset E^{\otimes 2} \oplus E \oplus \mathbb{R}$.
- Koszul dual $A^{¡} = \left(qA^{¡}, d, \theta\right)$ is a curved dg-coalgebra. To define it:
- Relations split as:
  $R \ni r = \underbrace{r_{(2)}}_{\in qR} + \underbrace{r_{(1)}}_{=d\left(r_{(2)}\right)} + \underbrace{r_{(0)}}_{=\theta\left(r_{(2)}\right)} \in E^{\otimes 2} \oplus E \oplus \mathbb{R}$
  - Quadratic part: $qA = T(E)/\left(qR\right)$ where $qR = proj_{E^{\otimes 2}}(R)$
  - Linear part: $d: qA^{¡} \to qA^{¡}$ is a coderivation which extends $r_{(2)} \mapsto -r_{(1)}$.
  - Constant part: $\theta: qA^{¡} \to \mathbb{R}$ satisfies $d^2 = (\theta \otimes id - id \otimes \theta)\Delta$ and $\theta d = 0$.
- Example:
  - $\mathrm{U}(\mathfrak{g}) = T_+(\mathfrak{g})/\left(xy - yx - [x, y]\right)$.
  - $qA = S(\mathfrak{g})$
  - $d\left(xy\right) = [x, y]$
  - We recognize $A^{¡} = \mathrm{C}_*^{\mathrm{CE}}(\mathfrak{g})$.

Koszul resolutions

- Bar/cobar Ω: $\{\text{curv. dg. coalg.}\} \rightleftarrows \{\text{semi. aug. alg}\}: B$, where

$\Omega C = \left(T_+\left(\Sigma^{-1}C\right), d_2 + d_1 + d_0\right), \qquad \mathrm{B}A = \left(T^c\left(\Sigma\bar{A}\right), d_2 + d_1, \theta\right)$

- Canonical resolution $\Omega \mathrm{B}A \to A$… but very big!
- Theorem [Polischuck–Positselski] If qA is Koszul then $\Omega A^{¡} \to A$ is a resolution.
- Example: $A = U(\mathfrak{g})$ then $qA = S(\mathfrak{g})$ is Koszul $\Rightarrow \Omega \mathrm{C}_*^{\mathrm{CE}}(\mathfrak{g}) \to U(\mathfrak{g})$ is a resolution.
- Goal: do this for more general types of algebras (e.g., Poisson algebras).

Quadratic operads

Koszul duals

- Quadratic operad: $\mathcal{P} = \mathrm{Op}(E)/(R)$ where $E \in \mathfrak{S}\mathrm{Seq}$ and $R \subset E \circ_{(1)} E$.
- Example: Com = $\mathrm{Op}(\mu)/\Big(\mu(\mu(x,y),z) = \mu\left(x, \mu(y,z)\right)\Big)$.
- Koszul dual cooperad: $\mathcal{P}^{¡} = \mathrm{Op}^c\left(\Sigma E, \Sigma^2 R\right)$. Koszul dual operad: $\mathcal{P}^! = \mathrm{Op}(E^*)/\left(R^\perp\right)$.
- Examples: $\mathrm{Ass}^! = \mathrm{Ass}$, $\mathrm{Com}^! = \mathrm{Lie}$, $\mathrm{Lie}^! = \mathrm{Com}$, $\mathrm{Pois}_n^! = \mathcal{S}^{n-1}\mathrm{Pois}_n$.
- Koszul complex: $\mathrm{K}_{\mathcal{P}} = \left(\mathcal{P} \circ_{(1)} \mathcal{P}^{¡}, d\right)$. Acyclic iff $\mathcal{P}$ is Koszul.

Koszul resolutions

- Bar/cobar Ω: {coaug. cooperads} $\rightleftarrows$ {aug. operads}: B, where
  $\Omega \mathcal{C} = \left(\mathrm{Op}^c\left(\Sigma^{-1}\bar{\mathcal{C}}\right), d\right), \qquad \mathrm{B}\mathcal{P} = \left(\mathrm{Op}\left(\Sigma\bar{\mathrm{A}}\right), d\right)$
- Canonical resolution $\Omega \mathrm{B}\mathcal{P} \to \mathcal{P}$… But very big!
- Theorem [Ginzburg–Kapranov, Getzler–Jones, Geztler] If $\mathcal{P}$ is quadratic and Koszul, then smaller resolution $\mathcal{P}_\infty \coloneqq \Omega \mathcal{P}^{¡} \to \mathcal{P}$.
- In that case, $\mathcal{P}_\infty$-algebras are called "homotopy $\mathcal{P}$-algebras" and have a nicer homotopy theory.
- Examples: $\mathrm{Ass}_\infty = \mathrm{A}_\infty$, $\mathrm{Com}_\infty = \mathrm{C}_\infty$, $\mathrm{Lie}_\infty = \mathrm{L}_\infty$…

Big resolutions of algebras over operads

- $\mathcal{P} = \mathrm{Op}(E)/(R)$ a quadratic operad.
- $\kappa: \mathcal{P}^{¡} \to \mathcal{P}$ the twisting morphism ("projection on generators")
- Bar/cobar adjunction Ω_κ: {coaug. $\mathcal{P}^{¡}$ coalg} $\rightleftarrows$ {aug. $\mathcal{P}$ alg}: B_κ, where
  $\Omega_\kappa C = \left(\mathcal{P}\left(\Sigma^{-1}\bar{C}\right), d\right), \qquad B_\kappa A = \left(\mathcal{P}^{¡}\left(\Sigma\bar{A}\right), d\right)$
- If $\mathcal{P}$ is Koszul, then canonical resolution of $\mathcal{P}$-algebras $\Omega_\kappa B_\kappa A \to A$… but very big!
- Example: $\mathfrak{g}$ a Lie algebra $\Rightarrow B_\kappa \mathfrak{g} = C_*^{\mathrm{CE}}(\mathfrak{g})$, $\Omega_\kappa B_\kappa \mathfrak{g} = (\mathbb{L}C_{*-1}^{\mathrm{CE}}(\mathfrak{g}), d)$.
- Goal: smaller resolutions of algebras

Monogenic algebras

Koszul duals

- **Monogenic $\mathcal{P}$-algebra** [Millès]: $A = \mathcal{P}(V)/(S)$ where $S \subset E(V)$ $\quad$ ($\mathcal{P} = \mathrm{Op}(E)/(R)$).
- Remark: if $\mathcal{P}$ is binary, then monogenic $\Leftrightarrow$ quadratic.
- Koszul dual coalgebra: $A^{¡} = \Sigma\mathcal{P}^{¡}(V, \Sigma S)$. Dual algebra: $A^! = \mathcal{P}^!(V^*)/\left(S^\perp\right)$.
- If $\mathcal{P}$ = Ass, then classical Koszul duals.
- Example for $\mathcal{P}$ = Com: $U\left(A^!\right) = \left(A_{\mathrm{Ass}}\right)^!$ [Löfwall].
- Topological example:
  - $\mathrm{A} = e_2^*(r) = S\left(\omega_{ij}\right)_{1\le i<j\le r}/\left(\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ik} + \omega_{ik}\omega_{ij}\right)$, then we get
  - $\mathrm{A}^! = \mathfrak{p}_2(r) = \mathbb{L}\left(t_{ij}\right)_{1\le i<j\le r}/\left(\left[t_{ij}, t_{kl}\right], \left[t_{ik}, t_{ij} + t_{jk}\right]\right)$.

Koszul resolutions

- $A = \mathcal{P}(V)/(S)$ monogenic $\mathcal{P}$-algebra $\Rightarrow$ Koszul complex $K_A = (A \otimes A^{¡}, d(\Sigma v) = v)$.

- Theorem [Millès]: If $\mathcal{P}$ is quadratic Koszul and A is monogenic Koszul, then $\Omega_\kappa A^{¡} \to A$ is a resolution of A.
- Get much smaller resolutions of algebras over operads in this way.
- But still restricted to non-unital case.

Unitary operads

Setting

- Inspiration: Hirsh-Millès' Koszul duality of operads with units.
- $\mathcal{P} = \mathrm{Op}(E)/(R)$ where $R \subset \left(E \circ_{(1)} E\right) \oplus E \oplus \mathbb{R}\mathrm{id}$ with two conditions:
  - Minimal generators: $R \cap (E \oplus \mathbb{R}\mathrm{id}) = 0$
  - Maximal relations: $R = (R) \cap \left(\left(E \circ_{(1)} E\right) \oplus E \oplus \mathbb{R}\mathrm{id}\right)$.
- Quadratic version: $q\mathcal{P} = \mathrm{Op}(E)/\left(qR\right)$ where $qR = \mathrm{proj}_{E^{\otimes 2}}(R)$.
- Examples:
  - $\mathrm{uCom} = \mathrm{Op}(\mu, \eta)/\left(\mu\left(x, \mu(y,z)\right) = \mu(\mu(x,y),z);\ \mu(x,\eta) = x\right)$.
  - $\mathrm{cLie} = \mathrm{Op}(\lambda, c)/\left(\lambda\left(x, \lambda(y,z)\right) + \lambda(\lambda(x,y),z) + \lambda(y,\lambda(x,z)); \lambda(x,c) = 0\right)$.

Koszul duals

- $\mathcal{P} = \mathrm{Op}(E)/(R)$, $E \subset \left(E \circ_{(1)} E\right) \oplus E \oplus \mathbb{R}\mathrm{id}$

  $\mathrm{R} \ni r = \underbrace{r_{(2)}}_{\in E \circ_{(1)} E} + \underbrace{r_{(1)}}_{\phi_1\left(r_{(2)}\right) \in E} + \underbrace{r_{(0)}}_{\phi_0\left(r_{(2)}\right) \in \mathbb{R}\mathrm{id}}$
- Curved Koszul dual dg-cooperad: $\mathcal{P}^{¡} = (q\mathcal{P}^{¡}, d, \theta)$
  - Quadratic version $q\mathcal{P} = \mathrm{Op}(E)/qR$ where $qR = \mathrm{proj}_{E^{\otimes 2}}(R)$
  - d: coderivation which extends $q\mathcal{P}^{¡} \to \Sigma^2 qR \xrightarrow{\phi_1} \Sigma E$
  - $\theta: q\mathcal{P}^{¡} \to \Sigma^2 qR \xrightarrow{\phi_0} \mathbb{R}$ is the curvature
  - $d^2 = \left(\mathrm{id} \circ_{(1)} \theta - \theta \circ_{(1)} \mathrm{id}\right)\Delta_{(1)}$, and $\theta d = 0$.
- Example: $qu\mathrm{Com} = \mathrm{Com} \times \mathbb{R}\eta$ ($\Rightarrow$ big Koszul dual!), $d = 0$, $\theta\left(\mu^c \circ_1 \eta^c\right) = -1$.

Koszul resolutions

- Bar/cobar adjunction Ω: $\{\text{curved dg. cooperads}\} \rightleftarrows \{\text{semi. aug. unit. operads}\}$: B, where

  $\Omega(\mathcal{C}, d, \theta) = \left(\mathrm{Op}\left(\Sigma^{-1}\bar{\mathcal{C}}\right), d_0 + d_1 + d_2\right), \qquad \mathrm{B}\mathcal{P} = \left(\mathrm{Op}^c\left(\Sigma\bar{\mathcal{P}}\right), d_1 + d_2, \theta\right)$
- Canonical resolution $\Omega B\mathcal{P} \to \mathcal{P}$... but very big!
- Theorem [Hirsh-Millès] If $q\mathcal{P}$ is Koszul, then $\mathcal{P}_\infty \coloneqq \Omega\left(q\mathcal{P}^{¡}, d, \theta\right) \to \mathcal{P}$ is a resolution.
- Much smaller than $\Omega B\mathcal{P}$. For example, $\mathcal{P} = u\mathrm{Ass}$: generators = A_∞ generators with some inputs plugged by unit, differential = A_∞ with plugged inputs distributed.

Curved Koszul duality of operadic algebras

Unital operads

- $\mathcal{P} = \mathrm{Op}(E)/(R)$ binary quadratic operad
- Unital version $u\mathcal{P} = \mathrm{Op}\left(E \oplus \eta\right)/(R + R')$ such that:
  - $E \hookrightarrow E \oplus \eta$ induces an injection $\mathcal{P} \hookrightarrow u\mathcal{P}$

    - $qu\mathcal{P} \cong \mathcal{P} \oplus \eta$
    - R' has only quadratic terms
- Examples: $u\mathrm{Com}, u\mathrm{Ass}, u\mathrm{Lie}$…

Unital algebras

- $u\mathcal{P}$-algebra with QLC relations:
    - $A = u\mathcal{P}(V)/I$
    - $I = (S)$ where $S = I \cap \big(\eta \oplus V \oplus E(V)\big)$
    - $S \cap \big(\eta \oplus V\big) = 0$
- In particular, $S = \big\{x + \alpha_0(x) + \alpha_1(x) \mid x \in qS\big\}$ where qS is the projection onto $E(V)$, $\alpha_0(x) \in \mathbb{R}\eta$ and $\alpha_1(x) \in V$.
- Automatically semi-augmented $\epsilon: A \to \mathbb{R}$

Curved Koszul dual

- $A^{¡} = (qA, d, \theta)$ with:
    - $qA = \mathcal{P}(V)/\big(qS\big)$ has Koszul dual in sense of Millès $qA^{¡} = \Sigma\mathcal{P}^{¡}\big(V, \Sigma qS\big)$
    - $d: qA^{¡} \to \Sigma\mathcal{P}^{¡}(V)$ is the unique coderivation that extends $\alpha_1 \circ \mathrm{proj}_{qS}$
    - Curvature $\theta = \alpha_0 \circ \mathrm{proj}_{qS}$
- Satisfies the following:
    - $d\big(qA^{¡}\big) \subset qA^{¡}$
    - Generalization of $(\mathrm{id} \otimes \theta - \theta \otimes \mathrm{id})\Delta = d^2$, concretely:
      $d^2 = \gamma \circ (\kappa \circ' \theta) \circ \mathrm{proj}_{(2)} \circ \Delta_C$

$$(3.3) \quad \star_\varphi(\theta) : C \xrightarrow{\Delta_C} \Sigma\mathsf{C}(\Sigma^{-1}C) \twoheadrightarrow \Sigma\mathsf{C}(2) \otimes_{\Sigma_2} (\Sigma^{-1}C)^{\otimes 2} \xrightarrow{\varphi\circ'\theta}$$
$$\xrightarrow{\varphi\circ'\theta} \Sigma^2 u\mathsf{P}(2) \otimes ((\Sigma\Bbbk^{\dagger} \otimes \Sigma^{-1}C) \oplus (\Sigma^{-1}C \otimes \Sigma\Bbbk^{\dagger})) \xrightarrow{\gamma_{u\mathsf{P}}} \Sigma^2 u\mathsf{P}(C),$$

Koszul resolutions

- Bar/cobar adjunction $\Omega_\kappa: \{\text{curved dg } \mathcal{P}^{¡} \text{ coalgebras}\} \rightleftarrows \{\text{semi aug } \mathcal{P} \text{ algebras}\}: \mathrm{B}_\kappa$
- Canonical resolution $\Omega_\kappa B_\kappa A \to A$… but very big!
- Koszul morphism $\Omega A^{¡} \to A$
- Theorem [I] If qA is Koszul in the sense of Millès, then $\Omega A^{¡} \to A$ is a resolution.

Applications

Factorization homology

- M framed n-manifold, A: algebra over uE_n
- Factorization homology of M with coefficients in A:
  $\int_{\mathrm{M}} \mathrm{A} = \mathrm{hocolim}_{(D^n)^{\sqcup k} \hookrightarrow M} A^{\otimes k}$
- Scary definition, but…
- Theorem [Ayala-Francis] $\int_M A = E_M \circ^{\mathbb{L}}_{uE_n} A = \mathrm{Tor}_{uE_n}\big(E_M, A\big)$
- Data is thus separated into three independent pieces + resolution

Formality

- Let us work over $\mathbb{R}$ and compute chains
- $\mathrm{C}_*\left(\int_M A\right) \simeq C_*\big(E_M\big) \circ^{\mathbb{L}}_{C_*(uE_n)} C_*(A)$ because chains preserve homotopy colimits
- Theorem [Kontsevich, Tamarkin, Lambrechts-Volić, Petersen, Fresse-Willwacher, Boavida-Horel]
  The operad uE_n is formal, i.e., $C_*\big(uE_n\big) \simeq H_*\big(uE_n\big)$
- $H_*\big(uE_n\big) = u\mathrm{Pois}_n$ encodes unital Poisson n-algebras

Lambrechts-Stanley model

- Let M be a simply connected closed smooth manifold, $n = \dim M \geq 4$
- $P \simeq \Omega^*(M)$ CDGA which satisfies Poincaré duality at the level of cochains
- Theorem [I]. Model of $E_M(r)$, compatible with operad structure:

$$\mathrm{G}_P(r) := \left(\frac{A^{\otimes r} \otimes S\left(\omega_{ij}\right)_{1\le i\neq j\le r}}{\omega_{ij}^2;\ \omega_{ji} - (-1)^n \omega_{ij};\ \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij};\ \left(p_i^*(x) - p_j^*(x)\right)\omega_{ij}}, d\omega_{ij} = \Delta_{ij} \right)$$

- Explicit description:
  $\mathrm{G}_P^\vee \cong \mathrm{C}_*^{\mathrm{CE}}\left(P^{n-*} \otimes \mathrm{Lie}_n[1-n]\right)$ + action of Com
- Upshot: $C_*\left(\int_M A\right) \simeq \mathrm{G}_P^\vee \circ^{\mathbb{L}}_{u\mathrm{Pois}_n} \tilde{A}$ where $\tilde{A}$ is a $u\mathrm{Pois}_n$-algebra representing A

Weyl n-algebras

- $A = \mathcal{O}_{\mathrm{poly}}\left(T^*\mathbb{R}^d[1-n]\right) = \mathbb{R}\left[x_1, \dots, x_d, \xi_1, \dots, \xi_d\right]$ with $\left\{x_i, \xi_j\right\} = \delta_{ij}1$
- This algebra has a QLC presentation as a $u\mathrm{Pois}_n$-algebra
- Quadratic version: $qA = \mathbb{R}\left[x_i, \xi_j\right]$ with trivial bracket. Clearly Koszul!
- Koszul dual $A^{¡} = \left(qA^{¡}, d, \theta\right)$:
  - $qA^{¡} = S^c\left(\bar{x}_i, \bar{\xi}_j\right)$ cofree symmetric coalgebra with trivial cobracket.
  - Differential $d = 0$ (no linear terms in relations)
  - Curvature $\theta\left(\bar{x}_i \wedge \bar{\xi}_j\right) = -\delta_{ij}$ and zero on other basis elements.
- Small resolution $Q_A = \Omega_\kappa A^{¡} = \left(SLS^c\left(\bar{x}_i, \bar{\xi}_j\right)\right) \to A$
- Much smaller than if we had applied resolutions of operads: $\Omega_\kappa B_\kappa A \supset SLS^c L^c S\left(x_i, \xi_j\right)$, plus resolution of the unit!
- Theorem [I, see also Markarian, Döppenschmidt]
  $\int_{\mathrm{M}} \mathcal{O}_{\mathrm{poly}}\left(T^*\mathbb{R}^d[1-n]\right) \simeq \mathrm{C}_*^{\mathrm{CE}}\left(P^{n-*} \otimes \left\langle x_i, \xi_j \right\rangle\right) \simeq 0$
- Result is not unexpected: for a quantum observable with values in A, the "expectation" lives in $\int_M A$, and it should be a single number for closed manifolds.