Configuration Spaces of Compact Manifolds

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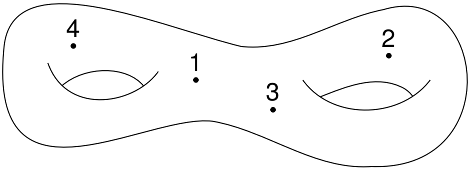
Introduction

M: n-manifold (+ adjectives) $\rightsquigarrow$ configuration spaces

$$\mathrm{Conf}_k(M) := \{(x_1, \dots, x_k) \in M^k \mid \forall i \neq j,\ x_i \neq x_j\}$$



Goal

Obtain a CDGA model of $\mathrm{Conf}_k(M)$ from a CDGA model of M

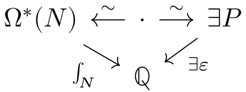
Closed manifolds: Poincaré duality models

Poincaré duality CDGA (P, d, ε) (example: $P = H^*(N)$ for N closed)

- (P, d): finite type connected CDGA;
- $\varepsilon : P^n \to \mathbb{Q}$ such that $\varepsilon \circ d = 0$;
- $P^k \otimes P^{n-k} \to \mathbb{Q}$, $a \otimes b \mapsto \varepsilon(ab)$ non degenerate.

Theorem (Lambrechts & Stanley 2008)

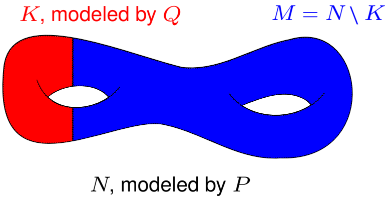
Any simply connected closed manifold has such a model.



Remark

Reasonable assumption: $\exists$ non simply-connected $L \simeq L'$ but $\mathrm{Conf}_k(L) \not\simeq \mathrm{Conf}_k(L')$ for $k \geq 2$ [Longoni & Salvatore].

Starting data:



- Poincaré duality CDGA P
- CDGA Q s.t. $Q^{\geq n/2-1} = 0$
- $\psi : P \twoheadrightarrow Q$

Yields $\psi^! : Q^\vee[-n] \to P^\vee[-n] \cong P$

Surjective pretty model:

$$\begin{array}{ccccc}
P \oplus_{\psi^!} Q^\vee[1-n] & \xleftarrow{\sim} & \cdot & \xrightarrow{\sim} & \Omega^*(M) \\
\downarrow\!\!\downarrow {\scriptstyle \psi\oplus\mathrm{id}} & & \downarrow & & \downarrow {\scriptstyle \mathrm{res}} \\
Q \oplus_{\psi\psi^!} Q^\vee[1-n] & \xleftarrow{\sim} & \cdot & \xrightarrow{\sim} & \Omega^*(\partial M)
\end{array}$$

$A := P/\operatorname{im}(\psi^!) \simeq \Omega^*(M)$, non-degen pairing with $\ker(\psi) \simeq \Omega^*(M, \partial M)$

Pretty models and nice models

Theorem (Lambrechts & Stanley, Cordova Bunlens & L. & S.)

M admits a pretty model if:

- M is closed ($Q = 0$)
- M and ∂M are 2-connected + technical condition
- M is a disk bundle of rank $2k$ over a Poincaré duality space
- $M = N \setminus \mathrm{Tub}(K)$ where N is closed and $2 \dim K + 3 \leq \dim N$

Rather restrictive. More general: nice model:

$$\begin{array}{ccccc} B & \xleftarrow{\sim} & \cdot & \xrightarrow{\sim} & \Omega^*(M) \\ \lambda \downarrow\!\!\downarrow & & \downarrow & & \downarrow \mathrm{res} \\ B_\partial & \xleftarrow{\sim} & \cdot & \xrightarrow{\sim} & \Omega^*(\partial M) \end{array}$$

if $A := B / \ker \theta \simeq \Omega^*(M)$ is *isomorphic* to $(\ker \lambda)^\vee[-n] \simeq \Omega^{n-*}(M, \partial M)$

Proposition

This exists if $\dim M \geq 7$ and M and ∂M are simply connected

Diagonal class

In cohomology, diagonal class (N is closed)

$$[N] \in H_n(N) \mapsto \delta_*[N] \in H_n(N \times N) \qquad \delta(x) = (x, x)$$
$$\leftrightarrow \Delta_N \in H^{2n-n}(N \times N)$$

Representative in a Poincaré duality model (P, d, ε):

$$\Delta_P = \sum (-1)^{|x_i|} x_i \otimes x_i^\vee \in (P \otimes P)^n$$

$\{x_i\}$: graded basis and $\varepsilon(x_i x_j^\vee) = \delta_{ij}$ (independent of chosen basis)
Let Δ_A be the class in $A = P/(\cdots) \simeq \Omega^*(M)$

The model

$\mathrm{Conf}_k(\mathbb{R}^n)$ is a formal space, with cohomology [Arnold, Cohen]:

$$H^*(\mathrm{Conf}_k(\mathbb{R}^n)) = S(\omega_{ij})_{1\le i\ne j\le k}/I, \quad \deg \omega_{ij} = n-1$$
$$I = \langle \omega_{ji} = \pm\omega_{ij},\ \omega_{ij}^2 = 0,\ \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0\rangle.$$

$\mathtt{G}_A(k)$ conjectured model of $\mathrm{Conf}_k(M) = M^{\times k} \setminus \bigcup_{i\ne j} \Delta_{ij}$

- "Generators": $A^{\otimes k} \otimes S(\omega_{ij})_{1\le i\ne j\le k}$
- Relations:
  - Arnold relations for the ω_{ij}
  - $p_i^*(a)\cdot\omega_{ij} = p_j^*(a)\cdot\omega_{ij}$. $\quad (p_i^*(a) = 1\otimes\cdots\otimes 1\otimes a\otimes 1\otimes\cdots\otimes 1)$
- $d\omega_{ij} = (p_i^*\cdot p_j^*)(\Delta_A)$.

First examples

$$\mathtt{G}_A(k) = (A^{\otimes k} \otimes S(\omega_{ij})_{1\le i<j\le k}/J, d\omega_{ij} = (p_i^* \cdot p_j^*)(\Delta_A))$$

$\mathtt{G}_A(0) = \mathbb{R}$: model of $\mathrm{Conf}_0(M) = \{\emptyset\}$ ✓

$\mathtt{G}_A(1) = A$: model of $\mathrm{Conf}_1(M) = M$ ✓

$$\begin{aligned}\mathtt{G}_A(2) &= \left(\frac{A\otimes A\otimes 1 \ \oplus \ A\otimes A\otimes\omega_{12}}{1\otimes a\otimes\omega_{12} \equiv a\otimes 1\otimes\omega_{12}}, d\omega_{12} = \Delta_A\otimes 1\right) \\ &\cong (A\otimes A\otimes 1 \ \oplus \ A\otimes_A A\otimes\omega_{12},\ d\omega_{12} = \Delta_A\otimes 1) \\ &\cong (A\otimes A\otimes 1 \ \oplus \ A\otimes\omega_{12},\ d\omega_{12} = \Delta_A\otimes 1) \\ &\xrightarrow{\sim} A^{\otimes 2}/(\Delta_A)\end{aligned}$$

Brief history of $\mathtt{G}_A$

1969 [Arnold & Cohen] $H^*(\mathrm{Conf}_k(\mathbb{R}^n)) = \mathtt{G}_{H^*(D^n)}(k)$

1978 [Cohen & Taylor] $E^2 = \mathtt{G}_{H^*(N)}(k) \implies H^*(\mathrm{Conf}_k(N))$

~1994 For smooth projective complex manifolds ($\implies$ Kähler):

- [Kříž] $\mathtt{G}_{H^*(N)}(k)$ model of $\mathrm{Conf}_k(N)$
- [Totaro] The Cohen–Taylor SS collapses

2004 [Lambrechts & Stanley] $P^{\otimes 2}/(\Delta_P)$ model of $\mathrm{Conf}_2(N)$ for a 2-connected manifold

~2004 [Félix & Thomas, Berceanu & Markl & Papadima] $\mathtt{G}^\vee_{H^*(M)}(k) \cong$ page E^2 of Bendersky–Gitler SS for $H^*(N^{\times k}, \bigcup_{i \neq j} \Delta_{ij})$

2008 [Lambrechts & Stanley] $H^*(\mathtt{G}_P(k)) \cong_{\Sigma_k-\mathsf{gVect}} H^*(\mathrm{Conf}_k(N))$

2015 [Cordova Bulens] $P^{\otimes 2}/(\Delta_P)$ model of $\mathrm{Conf}_2(N)$ for $\dim N = 2m$

2015 [CB–L–S] $\mathtt{G}_A(2)$ model of $\mathrm{Conf}_2(M)$ if M has a surjective pretty model

First part of Theorem A

Theorem

$\mathtt{G}_A(k)$ is a model over $\mathbb{R}$ of $\mathrm{Conf}_k(M)$ if M is simply connected, smooth, and

- $\partial M = \emptyset$ and $\dim M \geq 4$ [I., Campos & Willwacher], or
- M admits a surjective pretty model and $\dim M \geq 5$ [I. & Lambrechts], or
- M and ∂M are simply connected and $\dim M \geq 7$ [I. & Lambrechts].

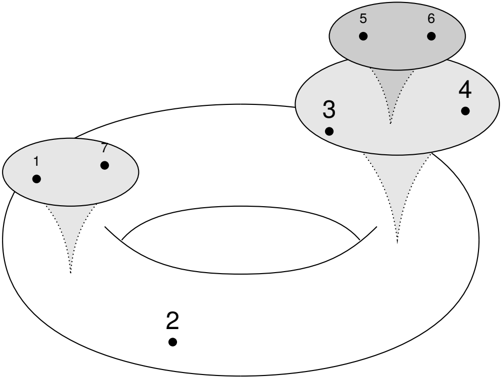
In all these cases, $(M, \partial M) \simeq (M', \partial M') \implies \mathtt{G}_A(k) \simeq \mathtt{G}_{A'}(k)$.

Idea of the proof

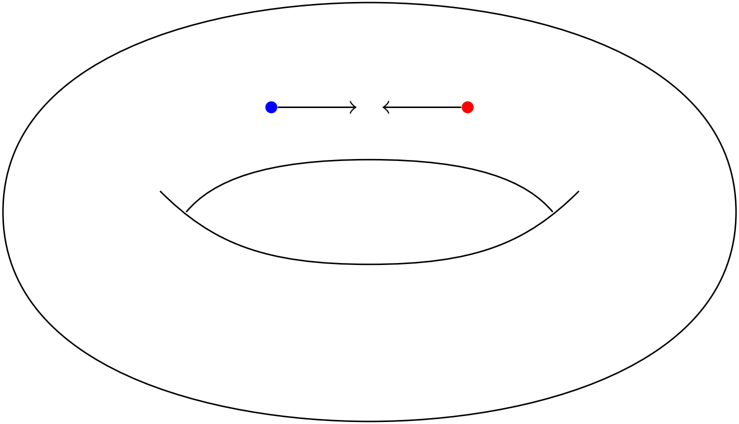
Idea

Study all of $\{\mathrm{Conf}_k(M)\}_{k\geq 0}$ at once: more structure! $\to$ module over an operad

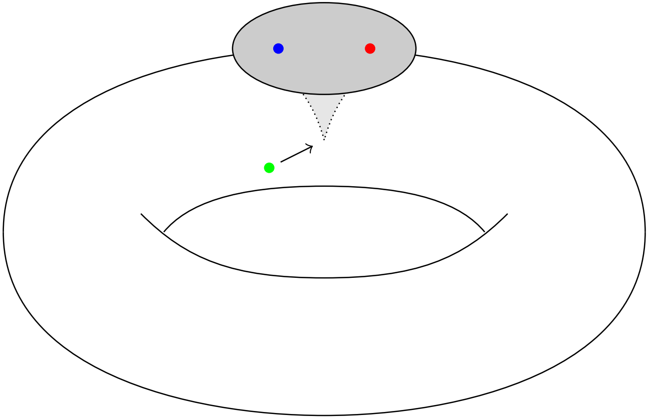
Fulton–MacPherson compactification $\mathrm{Conf}_k(M) \xrightarrow{\sim} \mathtt{FM}_M(k)$



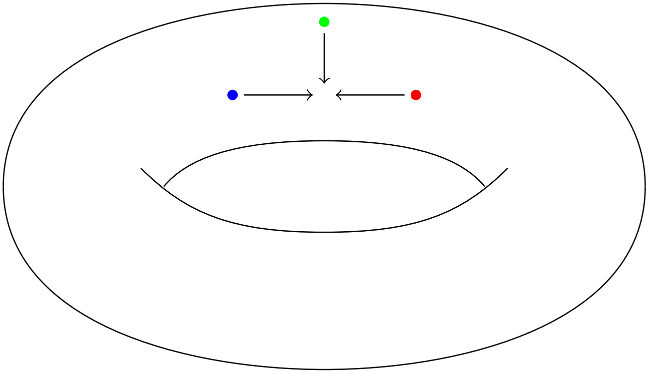
Animation #1



Animation #2

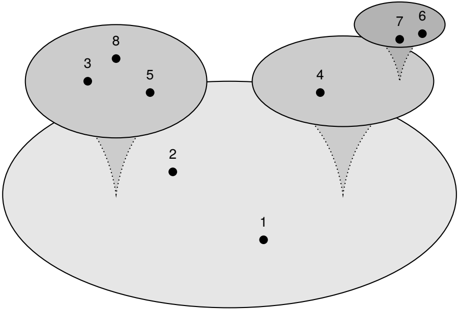


Animation #3



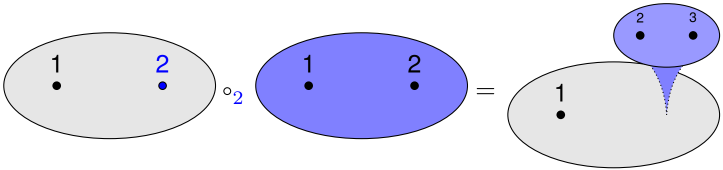
Compactifying $\mathrm{Conf}_k(\mathbb{R}^n)$

Can also compactify $\mathrm{Conf}_k(\mathbb{R}^n) \xrightarrow{\sim} \mathrm{Conf}_k(\mathbb{R}^n)/\mathrm{Aff}(\mathbb{R}^n) \overset{\sim}{\hookrightarrow} \mathtt{FM}_{\mathbb{R}^n}(k)$



(+ normalization to deal with $\mathbb{R}^n$ being noncompact)

$\mathtt{FM}_{\mathbb{R}^n} = \{\mathtt{FM}_{\mathbb{R}^n}(k)\}_{k \geq 0}$ is an operad: we can insert an infinitesimal configuration into another



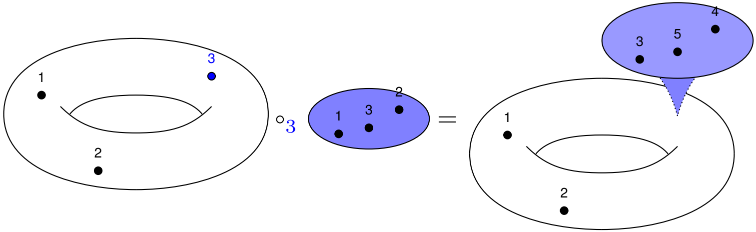
$$\mathtt{FM}_n(k) \times \mathtt{FM}_{\mathbb{R}^n}(l) \xrightarrow{\circ_i} \mathtt{FM}_n(k+l-1), \quad 1 \leq i \leq k$$

Remark

Weakly equivalent to the little n-disks operad.

Modules over operads

M framed $\implies \mathtt{FM}_M = \{\mathtt{FM}_M(k)\}_{k\geq 0}$ is a right $\mathtt{FM}_{\mathbb{R}^n}$-module: we can insert an infinitesimal configuration into a configuration on M



$$\mathtt{FM}_M(k) \times \mathtt{FM}_n(l) \xrightarrow{\circ_i} \mathtt{FM}_M(k+l-1), \quad 1 \leq i \leq k$$

Cohomology of $\mathtt{FM}_n$ and coaction on $\mathtt{G}_A$

$H^*(\mathtt{FM}_n)$ inherits a Hopf cooperad structure

One can rewrite:

$$\mathtt{G}_A(k) = (A^{\otimes k} \otimes H^*(\mathtt{FM}_n(k))/\text{relations}, d)$$

Proposition

$\chi(M) = 0$ or $\partial M \neq \emptyset \implies \mathtt{G}_A = \{\mathtt{G}_A(k)\}_{k \geq 0}$ is a Hopf right $H^*(\mathtt{FM}_n)$-comodule

We are looking for something to put here:

$$\mathtt{G}_A(k) \xleftarrow{\sim} \color{red}{?} \xrightarrow{\sim} \Omega^*(\mathtt{FM}_M(k))$$

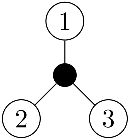
Hunch: if true, then hopefully it fits in something like this!

$$\begin{array}{ccccc} \mathtt{G}_A & \xleftarrow{\quad\sim\quad} & \color{red}{?} & \xrightarrow{\quad\sim\quad} & \Omega^*(\mathtt{FM}_M) \\ \circlearrowleft & & \color{red}{\circlearrowleft} & & \circlearrowleft \\ H^*(\mathtt{FM}_n) & \xleftarrow{\quad\sim\quad} & \color{red}{?} & \xrightarrow{\quad\sim\quad} & \Omega^*(\mathtt{FM}_n) \end{array}$$

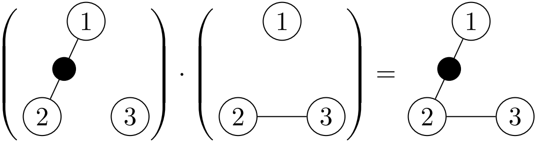
Fortunately, the bottom row is already known: formality of $\mathtt{FM}_n$

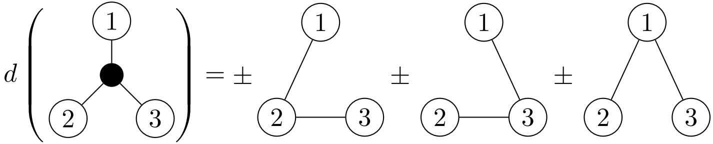
Kontsevich's graph complexes

[Kontsevich] Hopf cooperad $\mathtt{Graphs}_n = \{\mathtt{Graphs}_n(k)\}_{k\geq 0}$



$\in \mathtt{Graphs}_n(3)$

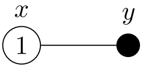




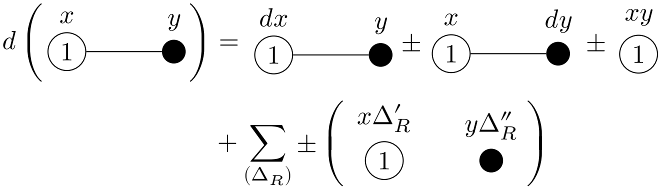
Theorem (Kontsevich 1999, Lambrechts–Volić 2014)

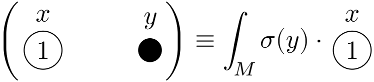
Labeled graph complexes

Labeled graph complex $\mathtt{Graphs}_R$:



$\in \mathtt{Graphs}_R(1)$ (where $x, y \in R$)





Complete version of Theorem A

Theorem (Complete version)

$$\begin{array}{ccccc} \mathtt{G}_A & \xleftarrow{\sim} & \mathtt{Graphs}_R & \xrightarrow{\sim} & \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M) \\ \circlearrowleft^{\dagger} & & \circlearrowleft^{\dagger} & & \circlearrowleft^{\ddagger} \\ H^*(\mathtt{FM}_n) & \xleftarrow{\sim} & \mathtt{Graphs}_n & \xrightarrow{\sim} & \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n) \end{array}$$

† When $\chi(M) = 0$ or $\partial M \neq \emptyset$

‡ When M is framed

Colored configuration spaces

When $\partial M \neq \emptyset$:

$$\begin{aligned} \mathrm{Conf}_{k,l}(M) &:= \{\underline{x} \in \mathrm{Conf}_{k+l}(M) \mid x_1, \dots, x_k \in \partial M, x_{k+1}, \dots, x_{k+l} \in \mathring{M}\} \\ &= \mathrm{Conf}_k(\partial M) \times \mathrm{Conf}_l(\mathring{M}) \end{aligned}$$

Remark

$\mathrm{Conf}_l(M)$ deformation retracts onto $\mathrm{Conf}_l(\mathring{M})$

$\implies$ can be compactified into $\mathtt{SFM}_M(k, l)$

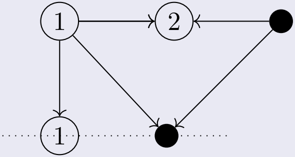
- points infinitesimally close to each other inside $\mathring{M}$
- points infinitesimally close to a point of ∂M

The Swiss-Cheese operad & graph complexes

Similar compactification $\mathtt{SFM}_n(k,l)$ of $\mathrm{Conf}_k(\mathbb{R}^{n-1} \times 0) \times \mathrm{Conf}_l(\mathbb{R}^{n-1} \times (0,+\infty))$
$\rightsquigarrow$ $\mathtt{SFM}_n$ "relative" operad over $\mathtt{FM}_n$

Theorem (Willwacher)

Graph complex model $\mathtt{SGraphs}_n \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathtt{SFM}_n)$:



$\in \mathtt{SGraphs}_n(1,2)$

Remarks

- if $n = 2$, a bit more complicated
- Swiss-Cheese is not formal [Livernet, Willwacher]
  $\implies \mathtt{SGraphs}_n \not\xrightarrow{\sim} H^*(\mathtt{SFM}_n)$

Model for colored configuration spaces

Straightforward generalization using labeled graphs:

Theorem (I. & Lambrechts)

M: smooth manifold with boundary satisfying the hypotheses of the previous theorem

$\implies$ model $(\mathtt{SGraphs}_R \curvearrowleft \mathtt{SGraphs}_n)$ of $(\Omega^*_{\mathrm{PA}}(\mathtt{SFM}_M) \curvearrowleft \Omega^*_{\mathrm{PA}}(\mathtt{SFM}_n))$

Thank you for your attention!

$\partial M = \emptyset$: arXiv:1608.08054
$\partial M \neq \emptyset$: https://idrissi.eu/pdf/thesis.pdf
These slides: https://idrissi.eu/talk/ethz2017/