Curved Koszul duality for algebras over unital operads

Najib Idrissi

May 2019 @ Higher Algebras in Topology, MPIM Bonn

Goal

Find resolutions of “algebras”.

Why?

- Compute derived invariants : derived tensor product, derived mapping space...
- Define homotopy algebras over operads.

Tool of choice: Koszul duality.

Starting data: quadratic algebra $A = T(E)/(R)$, $\quad R \subset E \otimes E$
$\rightsquigarrow$ Koszul dual $A^{¡}$: cofree coalgebra on ΣE with "corelations" $\Sigma^2 R$
(Usually easier to understand $A^! = F(E^*)/(R^\perp)$)

Examples

1. $A = T(E)$, $R = 0 \implies A^! = E^*$ with trivial multiplication;
2. $A = S(E) = T(E)/(xy - yx) \implies A^! = T(E^*)/(x^*y^* + y^*x^*) = \Lambda(E^*)$.

$\implies$ Koszul complex $K_A := (A \otimes A^{¡}, d_\kappa(\Sigma e) = e)$; A is Koszul if K_A is acyclic

Example

$T(E)$ and $S(E)$ are both Koszul.

Bar/cobar adjunction:

$$\Omega : \{\text{coaug.coalgebras}\} \leftrightarrows \{\text{aug.algebras}\} : B$$

where $BA = (T^c(\Sigma\bar{A}), d_B)$ and $\Omega C = (T(\Sigma^{-1}\bar{C}), d_\Omega)$.

Canonical morphism $\Omega BA \xrightarrow{\sim} A$ is always a cofibrant resolution...but big!

A quadratic $\implies \exists$ canonical morphism $\Omega A^{¡} \to A$

Theorem (Priddy '70s)

A is Koszul $\iff \Omega A^{¡} \xrightarrow{\sim} A$.

Much smaller resolution!

Examples

$A = T(E) \implies \Omega A^{¡} = A = T(E)$ versus $\Omega BA = TT^cF(E)$

$A = S(E) \implies \Omega A^{¡} = T\Lambda^c(E)$ versus $\Omega BA = TT^cS(E)$.

Quadratic-linear-constant algebra: $A = T_+(E)/(R)$ with $R \subset E^{\otimes 2} \oplus E \oplus \mathbb{R}1$

Koszul dual $A^{¡} = (qA^{¡}, d_{A^{¡}}, \theta_{A^{¡}})$: curved dg-coalgebra

$$r = \underbrace{r_{(2)}}_{\in qR} + \underbrace{r_{(1)}}_{d(r_{(2)})} + \underbrace{r_{(0)}}_{\theta(r_{(2)})} 1 \in R \subset E^{\otimes 2} \oplus E \oplus \mathbb{R}1.$$

- quadratic $\rightsquigarrow qA := T(E)/(qR)$ where $qR := \mathrm{proj}_{E^{\otimes 2}}(R)$;
- linear $\rightsquigarrow d_{A^{¡}} : qA^{¡} \to qA^{¡}$ is a coderivation;
- constant $\rightsquigarrow \theta_{A^{¡}} : qA^{¡} \to \mathbb{R}$ s.t. $d^2 = (\theta \otimes \mathrm{id} \mp \mathrm{id} \otimes \theta)\Delta$ and $\theta d = 0$.

Example

$A = U(\mathfrak{g}) = uF(\mathfrak{g})/(xy - yx - [x,y]) \rightsquigarrow qA = T(\mathfrak{g})/(xy - yx) = S(\mathfrak{g})$
$d_{A^{¡}}$= coderivation induced by $d(x \wedge y) = [x,y] \rightsquigarrow A^{¡} = C^{CE}_*(\mathfrak{g})$

Bar/cobar adjunction:

$$\Omega : \{\text{curved dg-coalgebras}\} \leftrightarrows \{\text{semi.aug.algebras}\} : B$$

where $BA = (T^c(\Sigma\bar{A}), d_2 + d_1, \theta)$ and $\Omega(C) = (T_+(\Sigma^{-1}C), d_2 + d_1 + d_0)$.

Theorem (Polischuck, Positselski)

If qA is Koszul then $\Omega A^{\text{i}} \xrightarrow{\sim} A$ is a cofibrant resolution.

Example

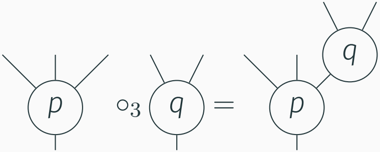
$A = U(\mathfrak{g}) \implies qA = S(\mathfrak{g})$ is Koszul $\implies \Omega C_*^{CE}(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g})$.

Goal: do this for more general types of unital algebras.

Operads

What are “more general types of algebras”?

Operad $\mathsf{P} = \{\mathsf{P}(n)\}_{n\geq 0}$: combinatorial object that encodes a type of algebra.



Examples

The “three graces”: $\mathtt{Ass}$ = associative algebras; $\mathtt{Com}$ = commutative algebras; $\mathtt{Lie}$ = Lie algebras.

E_n = homotopy associative and commutative (for $n \geq 2$) algebras.

$\mathsf{e}_n := H_*(\mathsf{E}_n) = \mathtt{Com} \circ \mathtt{Lie}_n$, $n \geq 2$ = Poisson n-algebras.

Quadratic operad: $\mathsf{P} = \mathrm{FOp}(E)/(R)$ where E is a generating set of operations and $R \subset E \circ_{(1)} E$ is a set of quadratic relations.

Example

$\mathtt{Com} = \mathrm{FOp}(\mu)/(\mu(\mu(x,y),z) = \mu(x,\mu(y,z)))$ is quadratic.

Formally similar definitions: Koszul dual cooperad $\mathsf{P}^{¡} = \mathrm{FOp}^{c}(\Sigma E, \Sigma^2 R)$ and its linear dual $\mathsf{P}^{!} = \mathrm{FOp}(E^*)/(R^{\perp})$.

Examples

$\mathtt{Ass}^{!} = \mathtt{Ass}$; $\mathtt{Com}^{!} = \mathtt{Lie}$, $\mathtt{Lie}^{!} = \mathtt{Com}$; $\mathtt{e}_n^{!} = \mathtt{e}_n\{-n\}$.

Formally similar definitions: bar/cobar adjunction

$$\Omega : \{\text{coaug.cooperads}\} \leftrightarrows \{\text{aug.operads}\} : B$$

Canonical morphism $\Omega B\mathsf{P} \xrightarrow{\sim} \mathsf{P}$ always a resolution, but very big

Theorem (Ginzburg–Kapranov '94, Getzler–Jones '94, Getzler '95…)

If P is quadratic and Koszul, then $\mathsf{P}_\infty := \Omega\mathsf{P}^{\text{¡}} \xrightarrow{\sim} \mathsf{P}$.

In this case, P_∞-algebras = "homotopy P-algebras".

Examples

$\mathtt{Ass}_\infty = A_\infty$-algebras, $\mathtt{Com}_\infty = C_\infty$-algebras, $\mathtt{Lie}_\infty = L_\infty$-algebras…

Big resolution of operadic algebras

$\mathsf{P} = \mathrm{FOp}(E)/(R)$ Koszul quadratic operad $\rightsquigarrow$ bar/cobar adjunction:

$$\Omega_\kappa : \{\text{coaug. } \mathsf{P}^{¡}\text{-coalgebras}\} \leftrightarrows \{\text{aug. } \mathsf{P}\text{-algebras}\} : B_\kappa,$$

where $\Omega_\kappa C = (\mathsf{P}(\Sigma^{-1}\bar{C}), d)$ and $B_\kappa A = (\mathsf{P}^{¡}(\Sigma\bar{A}), d)$.

$\rightsquigarrow$ resolution of P-algebras: $\Omega_\kappa B_\kappa(-)$, but very big.

Example

For a Lie algebra $\mathfrak{g}$, $\Omega_\kappa B_\kappa \mathfrak{g} = (L(C^{CE}_{*-1}(\mathfrak{g})), d)$.

KD for monogenic operadic algebras

Recall $\mathsf{P} = \mathrm{FOp}(E)/(R)$.
Monogenic P-algebras: $A = \mathsf{P}(V)/(S)$, $S \subset E(V)$.
(Monogenic = quadratic for binary P)

Koszul dual: $A^{¡} := \mathsf{P}^{¡}(\Sigma V, \Sigma^2 S)$, $A^! = \mathsf{P}(V^*)/(S^\perp)$.
Koszul complex: $K_A = (A \otimes A^{¡}, d_\kappa(\Sigma v) = v)$.

Theorem (Millès '12)

If P is quadratic Koszul and if A is a Koszul monogenic algebra, then $\Omega_\kappa A^{¡} \xrightarrow{\sim} A$ is a resolution of A.

Examples

$\mathsf{P} = \mathsf{Ass}$: recovers the classical Koszul duality of associative algebras.
A: quadratic Com-algebra $\implies U(A^!) = (A_{\mathsf{Ass}})^!$ [Löfwall].

Operads with QLC relations $uP = \mathrm{FOp}(E)/(R)$, $R \subset E \circ_{(1)} E \oplus E \oplus \mathbb{R}\,\mathrm{id}$

Koszul dual curved cooperad: $uP^{¡} = (quP^{¡}, d_{A^{¡}}, \theta_{A^{¡}})$

- quadratic $\rightsquigarrow$ quP: "quadratization" of uP;
- linear $\rightsquigarrow$ $d_{A^{¡}} : quP^{¡} \to quP^{¡}$ coderivation;
- constants $\rightsquigarrow$ $\theta_{A^{¡}} : quP^{¡} \to \mathbb{R}\,\mathrm{id}$ s.t. $d^2 = (\theta \circ \mathrm{id} \mp \mathrm{id} \circ \theta)\Delta$ and $\theta d = 0$

Example

$uCom = \mathrm{FOp}(\mu, \text{⫯})/(\mu(\mu(x,y),z) = \mu(x,\mu(y,z)),\ \mu(\text{⫯}, x) = x)$
$uCom^{¡} = (Com^{¡} \oplus \text{⫯}^c,\ d = 0,\ \theta(\mu^c \circ_1 \text{⫯}^c) = -1)$

Bar/cobar extends to the curved setting

Theorem (Hirsh–Millès '12)

If quP is Koszul, then $uP_\infty := \Omega(uP^{¡}) \xrightarrow{\sim} uP$: resolution of uP

Setting for curved KD

Consider $\mathsf{P} = \mathrm{FOp}(E)/(R)$: binary quadratic operad
$\rightsquigarrow$ unital version $u\mathsf{P} = \mathrm{FOp}(E \oplus \mathbb{1})/(R + R')$:

- $E \hookrightarrow E \oplus \mathbb{1}$ induces $\mathsf{P} \hookrightarrow u\mathsf{P}$
- $qu\mathsf{P} \cong \mathsf{P} \oplus \mathbb{1}$
- R' has only quadratic-constant terms

Examples

$u\mathtt{Ass}$, $u\mathtt{Com}$, $c\mathtt{Lie}$, $u\mathtt{e}_n$...

Algebra with QLC relations $A = u\mathsf{P}(V)/I$:

- I is generated by $S := I \cap (\mathbb{1} \oplus V \oplus E(V))$
- $S \cap (\mathbb{1} \oplus V) = 0$ ("V is minimal")

The second condition is difficult to check!

Curved KD for algebras over binary unital operads

$u\mathsf{P} = \mathrm{FOp}(E \oplus ⫯)/(R + R')$: unital version of quadratic $\mathsf{P} = \mathrm{FOp}(E)/(R)$
$A = u\mathsf{P}(V)/(S)$: algebra w/ QLC relations $S \subset E(V) \oplus V \oplus ⫯$

Koszul dual: curved $\mathsf{P}^{¡}$-coalgebra $A^{¡} = (qA^{¡}, d_{A^{¡}}, \theta_{A^{¡}})$

- quadratic $\rightsquigarrow$ $qA = \mathsf{P}(V)/(qS)$: “quadratization” of A;
- linear $\rightsquigarrow$ $d_{A^{¡}}$: coderivation;
- constant $\rightsquigarrow$ $\theta : qA^{¡} \to \mathbb{R}⫯$ (+ relations)

Generalization of bar/cobar adjunction:

$$\Omega_\kappa : \{\text{curved } \mathsf{P}^{¡}\text{-coalgebras}\} \leftrightarrows \{\text{semi.aug. } u\mathsf{P}\text{-algebras}\} : B_\kappa$$

Theorem (I. ’18)

If qA is Koszul then $\Omega_\kappa A^{¡} \xrightarrow{\sim} A$ is a resolution.

M: framed n-manifold, A: $u\mathsf{E}_n$-algebra ($\exists$ version for unframed manifolds.)

Goal

Compute $\int_M A = \operatorname{hocolim}_{(D^n)^{\sqcup k} \hookrightarrow M} A^{\otimes k}$.

Theorem (Francis 2015)

$\int_M A \simeq \mathsf{E}_M \circ^{\mathbb{L}}_{u\mathsf{E}_n} A = \operatorname{hocoeq}\big(\mathsf{E}_M \circ u\mathsf{E}_n \circ A \rightrightarrows \mathsf{E}_M \circ A\big)$, where:

$$u\mathsf{E}_n(k) = \mathrm{Emb}^{\mathrm{fr}}(\underbrace{\mathbb{R}^n \sqcup \dots \sqcup \mathbb{R}^n}_{k\times}, \mathbb{R}^n); \quad \mathsf{E}_M(k) = \mathrm{Emb}^{\mathrm{fr}}(\underbrace{\mathbb{R}^n \sqcup \dots \sqcup \mathbb{R}^n}_{k\times}, M).$$

Upshot: data is separated in three + resolution

Chains of factorization homology over $\mathbb{R}$

If we work over $\mathbb{R}$ and we just want chains:

$$C_*(\int_M A) \simeq C_*(\mathsf{E}_M) \circ^{\mathbb{L}}_{C_*(u\mathsf{E}_n)} C_*(A).$$

Theorem (Kontsevich '99; Tamarkin '03 ($n = 2$); Lambrechts–Volić '14; Petersen '14 ($n = 2$); Fresse–Willwacher '15)

The operad $C_*(u\mathsf{E}_n)$ is formal: $C_*(u\mathsf{E}_n) \simeq u\mathtt{e}_n := H_*(u\mathsf{E}_n) = \mathtt{Com} \circ \mathtt{Lie}_n$.

Theorem (I.)

M closed, simply connected, smooth, $\dim M \geq 4 \implies$
Lambrechts–Stanley model of $C_*(\mathsf{E}_M)$ as a right $C_*(u\mathsf{E}_n)$-module:

$$\mathsf{LS}_M = C_*^{CE}(\mathcal{M}^{n-*} \otimes \mathtt{Lie}_n[1-n]) + \text{ action of } \mathtt{Com}.$$

Upshot: $C_*(\int_M A) \simeq \mathsf{LS}_M \circ^{\mathbb{L}}_{u\mathtt{e}_n} \widetilde{A}$
$\implies$ we need to resolve A as a $u\mathtt{e}_n$-algebra.

Weyl algebra $\mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n])$

$A = \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n]) = S(x_1, \ldots, x_d, \xi_1, \ldots, \xi_d)$

Action of $u\mathsf{e}_n$: free symmetric algebra and $\{x_i, \xi_j\} = \delta_{ij}1$
$\implies$ quadratic-(linear-)constant presentation

Quadratization $qA = S(x_i, \xi_j)$ free symmetric algebra + zero bracket
Koszul dual: $A^{\mathrm{i}} = (qA^{\mathrm{i}}, d, \theta)$

- $qA^{\mathrm{i}} = S^c(\bar{x}_i, \bar{\xi}_j)$ cofree symmetric coalgebra + trivial cobracket
- $d = 0$
- curvature: $\theta(\bar{x}_i \wedge \bar{\xi}_j) = -\delta_{ij}$.

$\implies$ “small” resolution $Q_A := \Omega_\kappa A^{\mathrm{i}} = (SLS^c(\bar{x}_i, \bar{\xi}_j), d) \xrightarrow{\sim} A$

(If we had applied curved KD at the level of operads instead:
$\Omega_\kappa B_\kappa A \supset (\underbrace{SL}_{\text{cobar}}\underbrace{S^cL^c}_{\text{bar}}\underbrace{S(x_i, \xi_j)}_{A}, d)$, + resolution of the unit…)

Computation of $\int_M \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n])$

We can also compute

$$\int_M \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n]) \simeq \mathsf{LS}_M \circ_{u\mathsf{e}_n} (SLS^c(\bar{x}_i, \bar{\xi}_j), d)$$

Theorem (I. '18, see also Markarian '17, Döppenschmitt '18)

$\int_M \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n]) \simeq C_*^{CE}(\mathcal{M}^{n-*} \otimes \mathbb{R}\langle 1, x_i, \xi_j \rangle) \simeq \mathbb{R}.$

Intuition: quantum observable with values in $A \rightsquigarrow$ "expectation" lives in $\int_M A$, should be a number.

Operad P + P-algebra $A \implies$ notion of A-modules

Examples

$\mathsf{P} = \mathtt{Ass} \to (A, A)$ bimodules; $\mathsf{P} = \mathtt{Com} \to A$-modules; $\mathsf{P} = \mathtt{Lie} \to$ representations of the Lie algebra.

$\exists$ an associative algebra $U_{\mathsf{P}}(A)$ s.t. left $U_{\mathsf{P}}(A)$-modules = A-modules

Proposition

For $A = \mathscr{O}_{\mathrm{poly}}(T^*\mathbb{R}^d[1-n])$, the derived enveloping algebra $U^{\mathbb{L}}_{u\mathsf{e}_n}(A)$ is q.iso to the underived one.

Thank you for your attention!

These slides: `https://idrissi.eu`