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Formalité opéradique et homotopie des espaces de configuration

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Jury

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Résumé. Dans une première partie, nous étudions l'opérade $\mathtt{SC}_2$ «Swiss-Cheese» de Voronov, qui gouverne l'action d'une algèbre $\mathtt{D}_2$ sur une algèbre $\mathtt{D}_1$. Nous construisons un modèle en groupoïdes de cette opérade et nous décrivons les algèbres sur ce modèle de manière similaire à la description classique des algèbres sur $H_*(\mathtt{SC}_2)$. Nous étendons notre modèle en un modèle rationnel dépendant d'un associateur de Drinfeld, et nous le comparons au modèle qui existerait si l'opérade $\mathtt{SC}_2$ était formelle.

Dans une seconde partie, nous étudions les espaces de configurations des variétés compactes, lisses, sans bord et simplement connexes. Nous démontrons sur $\mathbb{R}$ une conjecture de Lambrechts–Stanley qui décrit un modèle de tels espaces de configurations, avec comme corollaire leur invariance homotopique réelle. En nous fondant sur la preuve par Kontsevich de la formalité des opérades $\mathtt{D}_n$, nous obtenons en outre que ce modèle est compatible avec l'action de l'opérade de Fulton–MacPherson quand la variété est parallélisée. Cela nous permet de calculer explicitement l'homologie de factorisation d'une telle variété.

Enfin, dans une troisième partie, nous élargissons ce résultat à une large classe de variétés à bord. Nous utilisons d'abord une dualité de Poincaré–Lefschetz au niveau des chaînes pour calculer l'homologie des espaces de configurations de ces variétés, puis nous reprenons les méthodes du second chapitre pour obtenir le modèle, qui est compatible avec l'action de l'opérade Swiss-Cheese $\mathtt{SC}_n$.
Mots-clés : opérades, espaces de configuration (topologie), variétés topologiques, topologie algébrique

Abstract. In a first part, we study Voronov's "Swiss-Cheese" operad $\mathtt{SC}_2$, which governs the action of a $\mathtt{D}_2$-algebra on a $\mathtt{D}_1$-algebra. We build a model in groupoids of this operad and we describe algebras over this model in a manner similar to the classical description of algebras over $H_*(\mathtt{SC}_2)$. We extend our model into a rational model which depends on a Drinfeld associator, and we compare this new model to the one that we would get if the operad $\mathtt{SC}_2$ were formal.

In a second part, we study configuration spaces of closed smooth simply connected manifolds. We prove over $\mathbb{R}$ a conjecture of Lambrechts–Stanley which describes a model of such configuration spaces, and we obtain as corollary their real homotopy invariance. Moreover, using Kontsevich's proof of the formality of the operads $\mathtt{D}_n$, we obtain that this model is compatible with the action of the Fulton–MacPherson operad when the manifold is framed. This allows us to explicitly compute the factorization homology of such a manifold.

Finally, in a third part, we expand this result to a large class of manifolds with boundary. We first use a chain-level Poincaré–Lefschetz duality result to compute the homology of the configuration spaces of these manifolds, then we reuse the methods of the second chapter to obtain our model, which is compatible with the action of the Swiss-Cheese operad $\mathtt{SC}_n$.
English title: *Operadic Formality and Configuration Spaces*

Keywords: operads, configuration spaces (topology), topological manifolds, algebraic topology

Organisation de cette thèse

Outre le Chapitre 0 qui sert d'Introduction, chacun des trois autres chapitres est tirée d'un article :

- le Chapitre 1 est l'article « Swiss-Cheese Operad and Drinfeld Center », publié au *Israel J. Math* en 2017 [Idr17] ;
- le Chapitre 2 constitue la prépublication « The Lambrechts–Stanley Model of Configuration Spaces » [Idr16] ;
- le Chapitre 3 est tiré d'un travail en cours, en collaboration avec Pascal Lambrechts.

Tous ces chapitres sont en anglais. Un résumé substantiel en français de cette thèse est disponible sur les pages qui suivent. Le Chapitre 1 est en grande partie indépendant des Chapitres 2 et 3 et peut être lu séparément.

Organization of this thesis

Besides the Introduction in Chapter 0, each of the other three chapters is drawn from an article:

- Chapter 1 is the article "Swiss-Cheese Operad and Drinfeld Center", published at the *Israel J. Math* in 2017 [Idr17];
- Chapter 2 is the preprint "The Lambrechts–Stanley Model of Configuration Spaces" [Idr16];
- Chapter 3 is based on a current work-in-progress joint with Pascal Lambrechts.

All of these chapters are in English. A substantial summary of this thesis in French can be found in the next pages. Chapter 1 is largely independent from Chapters 2 and 3 and can be read separately.

Résumé

Dans cette thèse, nous étudions le type d'homotopie des espaces de configuration de variétés en utilisant des idées venant de la théorie des opérades. Ces espaces de configuration consistent en des collections de points deux à deux distincts dans une variété donnée. Nous répondons aux problèmes de l'invariance homotopique et de la définition de modèles rationnels de ces espaces. Nous adaptons et généralisons des constructions de Kontsevich, qui donnent la formalité des espaces de configuration (compactifiés) des espaces euclidiens en tant qu'opérade [Kon99].

La notion d'opérade fut initialement introduite dans le but d'étudier les espaces de lacets itérés en théorie de l'homotopie [May72; BV73]. La théorie a connu une renaissance considérable au milieu des années 90 quand, inspirés par un article de Kontsevich [Kon93], Ginzburg et Kapranov [GK94] ont démontré dans des travaux fondateurs que certains phénomènes de dualité en algèbre pouvaient s'interpréter en termes d'opérades. Depuis, de nombreuses nouvelles applications des opérades ont été découvertes dans plusieurs domaines des mathématiques.

Introduction

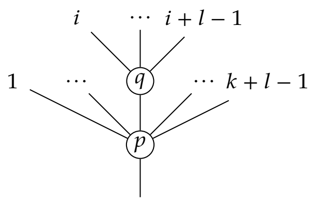
Opérades

Une opérade est un objet qui gouverne une catégorie d'algèbres. L'idée centrale de la théorie peut s'expliquer par analogie avec la théorie des représentations de groupes. Dans cette analogie, l'opérade correspond au groupe, et les algèbres sur l'opérade correspondent aux représentations du groupe.

Si un groupe est défini par une présentation par générateurs et relations, alors la catégorie de ses représentations peut se définir en termes des actions des générateurs sujettes aux relations. Dans notre analogie, une catégorie d'algèbres est définie par des opérations génératrices et des relations. Par exemple, la structure d'une algèbre associative se définit par une opération génératrice, le produit, et une relation, l'associativité. La structure d'une algèbre commutative se définit de façon similaire, avec la condition supplémentaire de symétrie du produit. Les algèbres de Lie sont définies par un crochet antisymétrique et la relation de Jacobi, etc. Nous pouvons interpréter chacune de ces définitions comme la définition d'une opérade par générateurs et relations qui gouvernerait la catégorie d'algèbres en question.

Tout comme nous étudions les groupes, il est intéressant d'étudier l'opérade elle-même, indépendamment de toute présentation par générateurs et relations. Cette nouvelle perspective nous permet de parler de morphismes, de sous-opérades, de quotients, d'extensions, etc., et de traduire des informations sur une opérade en des informations sur la catégorie des algèbres sur cette opérade.

Expliquons plus précisément ce qu'est une opérade.



Les structures algébriques associées aux opérades sont celles qui peuvent être décrites en termes d'opérations avec un nombre fini d'entrées et exactement une sortie. Une opérade[1] P est une collection $\mathsf{P} = \{\mathsf{P}(k)\}_{k\geq 0}$ « d'opérations » abstraites. On peut voir un élément de $\mathsf{P}(k)$ comme une opération à k entrées et une sortie. Le groupe symétrique Σ_k agit sur $\mathsf{P}(k)$, ce qui correspond à la permutation des entrées d'une opération. Il faut également se donner des opérations d'insertion

$$\circ_i : \mathsf{P}(k) \otimes \mathsf{P}(l) \to \mathsf{P}(k+l-1), \quad 1 \leq i \leq k,$$

qui modélisent la composition des opérations (tout comme la multiplication dans un groupe G correspond à la composition des actions sur les représentations). Enfin, une identité $\mathrm{id} \in \mathsf{P}(1)$ est un élément neutre pour la composition.

Pour illustrer cette définition, considérons l'exemple prototypique, l'*opérade des endomorphismes*. Étant donné un objet X dans une catégorie monoïdale symétrique, l'opérade des endomorphismes End_X est définie par $\mathsf{End}_X(n) := \mathrm{Hom}(X^{\otimes n}, X)$. L'action du groupe symétrique permute les entrées :

$$(f \cdot \sigma)(x_1, \dots, x_n) := f(x_{\sigma(1)}, \dots, x_{\sigma(n)}),$$

et les opérations d'insertion sont données par la composition des morphismes :

$$(f \circ_i g)(x_1, \dots, x_{k+l-1}) := f(x_1, \dots, x_{i-1}, g(x_i, \dots, x_{i+l-1}), x_{i+l}, \dots, x_{k+l-1}).$$

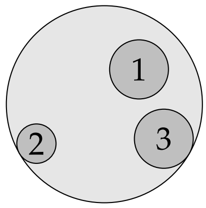
Enfin, l'identité $\mathrm{id} \in \mathsf{End}_X(1)$ est simplement l'identité de X. Une algèbre sur une opérade P est, par définition, un morphisme d'opérades $\mathsf{P} \to \mathsf{End}_X$, exactement comme une représentation d'un groupe G est la donnée d'un morphisme de monoïdes $G \to \mathrm{End}(V)$. Comme autre exemple, une opérade n'ayant que des opérations d'arité 1 est exactement la même chose qu'un monoïde, et une algèbre sur une telle opérade n'est autre qu'une représentation du monoïde associé. Nous renvoyons aux livres [LV12; Fre17] pour un traitement plus détaillé des opérades.

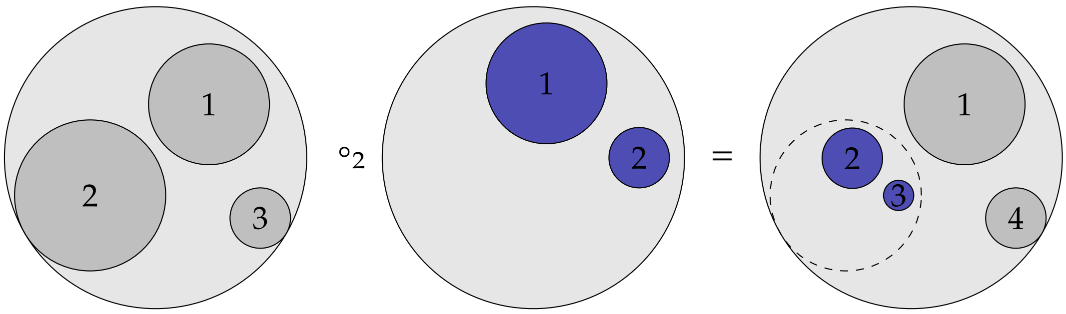
[1] Pour nous, une « opérade » sans autre qualificatif est une opérade symétrique à une couleur. Il existe d'autres variantes : opérades non symétriques, opérades cycliques, opérades colorées, etc.

Opérades des petits disques

Une famille d'opérades topologiques est d'un intérêt particulièrement grand : les opérades des petits n-disques D_n (ou, de manière équivalente, les opérades des petits n-cubes). Ce sont les opérades qui ont fait leur apparition dans l'étude originelle des espaces de configuration.

Un élément de $\mathsf{D}_n(k)$ est une configuration ordonnée de k petits n-disques aux intérieurs disjoints dans le disque unité D^n. Chaque disque de la configuration s'obtient comme l'image d'un plongement de D^n dans lui-même obtenu par la composition d'une translation et d'une homothétie. L'ensemble $\mathsf{D}_n(k)$ est muni de la topologie compacte-ouverte des plongements. L'action du groupe symétrique réordonne les disques d'une configuration, et l'insertion est donnée par la composition des plongements.





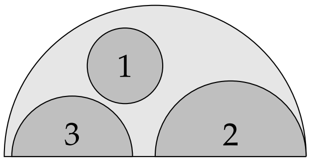
Un espace de lacets itéré $\Omega^n X$ est, presque par définition, une algèbre sur D_n. Le « principe de reconnaissance » [May72 ; BV73] dit que la réciproque est vraie : sous des hypothèses techniques, une D_n-algèbre « group-like » est faiblement équivalente à un espace de n-lacets.

Les opérades des petits disques se sont révélées, depuis cette première application, être utiles dans de nombreux contextes. Mentionnons la conjecture de Deligne [KS00 ; MS02], qui dit que les cochaînes de Hochschild $C^*(A; A)$ d'une algèbre associative sont munies d'une action de D_2 ; le théorème de formalité des cochaînes de Hochschild et ses applications à la quantification des variétés de Poisson [Kon99 ; Tam98 ; Kon03] ; le calcul de Goodwillie–Weiss et le calcul des espaces de plongements et des espaces de longs nœuds [Sin06 ; LTV10 ; AT14 ; DH12 ; BW13], ainsi que l'homologie de factorisation, en quelque sorte la « version covariante » du calcul des plongements [BD04 ; Lur09 ; Lur16 ; AF15 ; CG17].

Un résultat fondamental au sujet des opérades des petits disques est qu'elles sont *formelles* sur $\mathbb{Q}$ [Kon99 ; Tam03 ; LV14 ; FW15], c.-à-d. que l'opérade des chaînes $C_*(\mathsf{D}_n; \mathbb{Q})$ est quasi-isomorphe à son homologie $\mathsf{e}_n := H_*(\mathsf{D}_n; \mathbb{Q})$. Il suffit donc, sur $\mathbb{Q}$, d'étudier les opérades $\mathtt{e}_n$, qui ont une description combinatoire simple : $\mathtt{e}_1 = \mathtt{Ass}$ gouverne les algèbres associatives, et pour $n \geq 2$, $\mathtt{e}_n$ gouverne les $(n-1)$-algèbres de Poisson [Coh76]. Quand $n = 2$, la formalité de l'opérade $\mathtt{D}_2$ dépend du choix d'un associateur de Drinfeld, un objet qui définit une manière universelle de construire une catégorie monoïdale tressée à partir de données qui viennent de la théorie de Lie (voir p.ex. [Fre17, Chapter I.10] pour plus de détails).

Opérades « Swiss-Cheese »

Les opérades colorées (aussi connues sous le nom de multicatégories) généralisent les opérades et sont utilisées pour décrire des structures algébriques à plusieurs objets. Dans ce contexte, les entrées et la sortie d'une opération sont toutes étiquetées par une « couleur », et l'insertion n'est définie que si les couleurs correspondent. Dans la définition d'une algèbre sur une opérade colorée, chaque couleur correspond à un objet donné.



L'opérade « Swiss-Cheese » $\mathtt{SC} = \mathtt{SC}_2$ [Vor99] est une opérade à deux couleurs qui gouverne l'action d'une algèbre $\mathtt{D}_2$ sur une algèbre $\mathtt{D}_1$ par un morphisme central. Une opération de $\mathtt{SC}$ est donné par le plongement de disques et de demi-disques dans le demi-disque unité supérieur. Ces opérades sont liées aux OCHA de Kajiura et Stasheff [KS06b; Hoe09], et il existe une version Swiss-Cheese de la conjecture de Deligne [DTT11]. Des variantes en dimension supérieur $\mathtt{SC}_n$ gouvernent l'action d'une algèbre $\mathtt{D}_n$ sur une algèbre $\mathtt{D}_{n-1}$. Contrairement aux opérades des petits disques, les opérades Swiss-Cheese ne sont pas formelles [Liv15].

Espaces de configuration

Soit M une variété. Son $k$ième espace de configuration est donné par :

$$\mathrm{Conf}_k(M) := \{x \in M^k \mid \forall i \neq j,\ x_i \neq x_j\}.$$

Les espaces de configuration sont intimement liés aux opérades des petits disques. Par exemple, l'application $\mathtt{D}_n(k) \to \mathrm{Conf}_k(\mathbb{R}^n)$ qui associe à une configuration de disques la configuration constituée des centres des disques est une équivalence d'homotopie.

Cette construction n'est évidemment pas un invariant d'homotopie pour les variétés ouvertes dès que $k \geq 2$. Par exemple, pour $n \in \mathbb{N}$, $\mathrm{Conf}_2(\mathbb{R}^n) \simeq S^{n-1}$. Même en se restreignant aux variétés compactes sans bord, $\mathrm{Conf}_k(-)$ n'est pas un invariant d'homotopie [LS05b]. Le contre-exemple (donné par des espaces lenticulaires) n'est pas simplement connexe, et la question de savoir si $\mathrm{Conf}_k(-)$ est un invariant d'homotopie pour les variétés compactes, sans bord et simplement connexes reste ouverte.

Quelques résultats sont connus. Le type d'homotopie de $\Omega\,\mathrm{Conf}_k(M)$ ne dépend que de celui de $(M, \partial M)$ pour les variétés compactes connexes [Lev95]. Les espaces de configuration sont des invariants *stables* d'homotopie des variétés compactes sans bord [AK04]. Si M est une variété projective lisse, alors le type d'homotopie rationnel de $\mathrm{Conf}_k(M)$ ne dépend que de celui de M. Le même résultat est valable avec $k = 2$ pour les variétés compactes sans bord qui sont soit 2-connexes [LS04], soit simplement connexes et de dimension paire [Cor15].

Résultats du Chapitre 1

Comme mentionné précédemment, l'opérade Swiss-Cheese $\mathsf{SC} = \mathsf{SC}_2$ n'est pas formelle. En d'autres termes, il n'est pas possible de récupérer le type d'homotopie de SC à partir de son homologie $\mathsf{sc} = H_*(\mathsf{SC})$. Le but du premier chapitre de cette thèse est de trouver un modèle de SC qui corrige ce défaut de formalité.

Les trois opérades D_1, D_2 et SC sont asphériques, c.-à-d. que l'homotopie de chacun des espaces qui les composent est concentrée en degrés 0 et 1. Il y a donc, pour chacune de ces opérades (notons les P), une équivalence d'homotopie canonique $\mathsf{P} \xrightarrow{\sim} B\pi\mathsf{P}$, où B est le foncteur « espace classifiant » et π le foncteur « groupoïde fondamental » (les deux étant fortement monoïdaux, ils préservent la structure d'opérade). À homotopie près, il suffit donc d'étudier le groupoïde fondamental $\pi\mathsf{P}$ de chacune de ces trois opérades $\mathsf{P} \in \{\mathsf{D}_1, \mathsf{D}_2, \mathsf{SC}\}$ à « équivalence catégorique » (morphisme d'opérade induisant une équivalence de catégorie en chaque arité) près.

Il existe des descriptions classiques de modèles en groupoïdes des deux opérades D_1 et D_2.

- En chaque arité, $\mathsf{D}_1(r)$ est discret à homotopie près (avec $r!$ composantes connexes), et il est facile de voir que l'opérade $\pi\mathsf{D}_1$ est faiblement équivalente à l'opérade PaP des permutations parenthésées. La catégorie $\mathsf{PaP}(r) = \Sigma_r$ est discrète, avec comme objets les permutations de $\{1, \dots, r\}$, et la structure d'opérade est donnée par la composition par blocs des permutations.

- Les composantes $\mathsf{D}_2(r)$ de l'opérade D_2 sont des espaces classifiants des groupes de tresses pures P_r. L'opérade $\pi\mathsf{D}_2$ est ainsi faiblement équivalente à l'opérade PaB des tresses parenthésées (cf. [Fre17, Chapter I.3], voir aussi [Fie96]). Les objets de $\mathsf{PaB}(r)$ sont encore les permutations de $\{1, \dots, r\}$. Les morphismes entre deux permutations $\sigma, \sigma' \in \mathsf{PaB}(r)$ sont les tresses

> colorées à r brins entre ces deux permutations : chacun des brins de la tresse est « coloré » par un des nombres $\{1, \dots, r\}$, et l'ordre des brins au début (resp. à la fin) de la tresse est donné par σ (resp. σ'). La composition des tresses est donnée par l'insertion d'une tresse dans un voisinage tubulaire d'un brin.

Le premier résultat de ce premier chapitre (Theorem A) est la définition d'une opérade en groupoïdes PaPB des tresses et permutations parenthésées. Cette opérade combine, en quelque sorte, les deux opérades PaP et PaB pour obtenir une opérade colorée, et est faiblement équivalente au groupoïde fondamental $\pi\mathsf{SC}$.

La définition de PaPB est motivée par le résultat suivant. L'homologie de l'opérade Swiss-Cheese se scinde en un « produit de Voronov » $\mathtt{sc} = \mathtt{e}_2 \otimes \mathtt{e}_1$, où $\mathtt{e}_1 = H_*(\mathsf{D}_1) = \mathtt{Ass}$ gouverne les algèbres associatives et $\mathtt{e}_2 = H_*(\mathsf{D}_2) = \mathtt{Ger}$ gouverne les algèbres de Gerstenhaber [Vor99]. Concrètement, cela signifie qu'une algèbre sur $\mathtt{sc}$ est la donnée d'un triplet (A, B, f) où A est une algèbre associative, B est une algèbre de Gerstenhaber, et $f : B \to Z(A)$ est un morphisme d'algèbres de B dans le centre de A.

Les algèbres (dans la catégorie des catégories) sur l'opérade $\mathsf{PaP} \simeq \pi\mathsf{D}_1$ sont les catégories monoïdales, et les algèbres sur $\mathsf{PaB} \simeq \pi\mathsf{D}_2$ sont les catégories monoïdales tressées. En analogie avec le théorème de Voronov, nous démontrons que les algèbres (toujours dans la catégorie des catégories) sur $\mathsf{PaPB} \simeq \pi\mathsf{SC}$ sont les triplets $(\mathsf{M}, \mathsf{N}, F)$ où M est une catégorie monoïdale, N est une catégorie monoïdale tressée, et $F : \mathsf{M} \to \mathcal{Z}(\mathsf{N})$ est un foncteur monoïdal tressé de M dans le centre de Drinfeld de N – un analogue catégorique du centre d'une algèbre associative. Ce résultat est la contrepartie pour les opérades en groupoïdes de résultats ∞-catégoriques sur les algèbres à factorisation du demi-plan supérieur (cf. [Gin15, Proposition 31] et [AFT17, Example 2.13])

Dans une deuxième étape, nous fixons un associateur de Drindeld, que nous pouvons voir comme une équivalence rationnelle (au sens de la théorie de l'homotopie rationnelle) $\mathsf{PaB}_+ \to \widehat{\mathsf{CD}}_+$ où $\widehat{\mathsf{CD}}_+ = \mathrm{B}\,\hat{\mathbb{U}}\hat{\mathfrak{p}}_+$ est l'opérade complétée des diagrammes de cordes. À partir de cette donnée, nous construisons une opérade $\mathsf{PaP}\widehat{\mathsf{CD}}_+^{\varphi}$ rationnellement équivalente à la complétion de l'opérade Swiss-Cheese. Cette opérade étend la construction utilisée par Tamarkin [Tam03] pour démontrer la formalité de l'opérade D_2 et peut s'interpréter informellement comme un produit de Voronov tordu d'un modèle pour D_2 et d'un modèle pour D_1.

Résultats du Chapitre 2

Dans le second chapitre, nous étudions les espaces de configuration des variétés compactes sans bord simplement connexes. Dans la suite, on note M une telle variété. Nous nous plaçons dans le cadre de la théorie de l'homotopie rationnelle de Sullivan, où les espaces sont modélisés par des algèbres différentielles graduées commutatives (CDGA en anglais) et nous cherchons à obtenir un modèle de $\mathrm{Conf}_k(M)$ pour $k \geq 0$.

La cohomologie d'une variété orientée compacte sans bord satisfait la dualité de Poincaré, il y a un accouplement non dégénéré $H^k(M) \otimes H^{n-k}(M) \to \mathbb{Q}$ donné par $\alpha \otimes \beta \mapsto \alpha\beta \frown [M]$. Lambrechts et Stanley [LS08b] démontrent que si la variété est simplement connexe, alors cette propriété peut se traduire sur les modèles : M admet un modèle rationnel A à « dualité de Poincaré », c.-à-d. il y a un accouplement non-dégénéré $A^k \otimes A^{n-k} \to \mathbb{Q}$ induit par une « augmentation » $\varepsilon_A : A^n \to \mathbb{Q}$. À partir d'un tel modèle à dualité de Poincaré, ils introduisent une CDGA $\mathtt{G}_A(k)$ et démontrent que les nombres de Betti rationnels de $\mathtt{G}_A(k)$ coïncident avec ceux de $\mathrm{Conf}_k(M)$ [LS08a]. Ils conjecturent que cette CDGA est en fait un modèle rationnel de $\mathrm{Conf}_k(M)$, ce qui est connu pour les variétés projectives lisses complexes [Kri94], ainsi que quand $k = 2$ et que M est 2-connexe [LS04] ou de dimension paire [Cor15].

Cette CDGA avait déjà été étudiée dans certains cas particuliers [CT78 ; BG91 ; Kri94 ; Tot96 ; FT04]. Elle admet une description simple pour les premières valeurs de k : $\mathtt{G}_A(0) = \mathbb{Q}$ et $\mathtt{G}_A(1) = A$, ce qui est cohérent avec le fait que $\mathrm{Conf}_0(M) = \{*\}$ et $\mathrm{Conf}_1(M) = M$, et $\mathtt{G}_A(2)$ est quasi-isomorphe au quotient de $A^{\otimes 2}$ par l'idéal engendré par la « classe diagonale » de A (le dual de Poincaré de $M \subset M \times M$). Plus généralement, $\mathtt{G}_A(k)$ est en quelque sorte une version « étiquetée par A » de la description classique de la cohomologie de $\mathrm{Conf}_k(\mathbb{R}^n)$ [Arn69 ; Coh76].

La formalité de l'opérade des petits disques entraîne en particulier que les espaces de configuration $\mathrm{Conf}_k(\mathbb{R}^n)$ sont formels : la CDGA $H^*(\mathrm{Conf}_k(\mathbb{R}^n)) =: \mathtt{e}_n^\vee(k)$ (avec la différentielle nulle) est un modèle rationnel de $\mathrm{Conf}_k(\mathbb{R}^n)$. La preuve par Kontsevich de cette formalité fournit des morphismes explicites [Kon99 ; LV14]. L'idée de ce chapitre est d'adapter cette preuve aux espaces de configuration de M pour obtenir le fait que $\mathtt{G}_A(k)$ est un modèle.

Cette preuve fait intervenir la compactification de Fulton–MacPherson des espaces de configuration [FM94 ; AS94]. Étant donnée une variété M, l'espace $\mathtt{FM}_M(k)$ est une variété stratifiée dont l'intérieur est $\mathrm{Conf}_k(M)$. Le bord de $\mathtt{FM}_M(k)$ est obtenu en autorisant les points d'une configuration à devenir infinitésimalement proches les uns des autres. Il est possible de compactifier $\mathrm{Conf}_k(\mathbb{R}^n)$ de manière similaire en un espace $\mathtt{FM}_n(k)$, en tenant compte du fait que $\mathbb{R}^n$ lui-même n'est pas compact.

La preuve passe par la définition d'un certain complexe de graphes intermédiaires $\mathsf{Graphs}_n(k)$. On peut voir $H^*(\mathrm{Conf}_k(\mathbb{R}^n)) = H^*(\mathsf{FM}_n(k))$ comme un quotient de $\mathsf{Graphs}_n(k)$ par un idéal acyclique. Dans l'autre direction, il est nécessaire d'introduire le complexe des formes semi-algébriques par morceaux $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(k))$, qui est un modèle sur $\mathbb{R}$ de $\mathsf{FM}_n(k)$. Un quasi-isomorphisme de $\mathsf{Graphs}_n(k)$ dans $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(k))$ est alors donné par des intégrales le long des fibres des projections canoniques $\mathsf{FM}_n(k+l) \to \mathsf{FM}_n(k)$. Nous adaptons toutes ces constructions à $\mathrm{Conf}_k(M)$ pour obtenir un zigzag explicite de quasi-isomorphismes, pour M une variété lisse compacte sans bord simplement connexe de dimension au moins 4 :

$$\mathsf{G}_A(k) \xleftarrow{\sim} \mathsf{Graphs}_R(k) \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M(k)),$$

ce qui démontre que $\mathsf{G}_A(k)$ est un modèle sur $\mathbb{R}$ de $\mathsf{FM}_M(k) \simeq \mathrm{Conf}_k(\mathbb{R}^n)$ (première partie du Theorem C). Le point clé de la preuve consiste à démontrer que pour les variétés simplement connexes de dimension ≥ 4, la fonction de partition Z_φ, définie à l'aide d'intégrales sur FM_M, est (presque) triviale, ce qui s'obtient à l'aide d'un argument de comptage de degré.

Si A et A' sont deux modèles de M, nous démontrons directement que $\mathsf{G}_A(k) \simeq_{\mathbb{R}} \mathsf{G}_{A'}(k)$, d'où l'invariance homotopique réelle de $\mathrm{Conf}_k(M)$ par rapport à M (Corollary 2.4.36).

On peut insérer une configuration de FM_n dans une autre, ce qui donne une structure d'opérade faiblement équivalent à l'opérade des petits disques D_n. De même, quand M est parallélisée, on peut insérer une configuration de FM_n dans une configuration de FM_M et obtenir ainsi une structure de FM_n-module à droite sur FM_M. Quand $\chi(M) = 0$ (en particulier quand M est parallélisée), il est facile d'observer que la collection $\mathsf{G}_A = \{\mathsf{G}_A(k)\}$ est dotée d'une structure de comodule de Hopf à droite sur $H^*(\mathsf{FM}_n) = \mathsf{e}_n^\vee$. Nous démontrons que le zigzag de quasi-isomorphismes que nous construisons est compatible avec la structure de comodule quand M est parallélisée (deuxième partie du Theorem C). En d'autres termes, nous obtenons un modèle réel du FM_n-module à droite FM_M.

Cela nous permet de calculer l'homologie de factorisation, un invariant des variétés. Étant données une n-variété parallélisée M et une FM_n-algèbre B, l'homologie de factorisation de M à coefficients dans B peut se calculer comme le produit de composition relatif dérivé (cf. [AF15] et [Tur13, Section 5.1]) :

$$\int_M B := \mathsf{FM}_M \circ^{\mathbb{L}}_{\mathsf{FM}_n} B.$$

Notre résultat implique que si M est une variété lisse compacte sans bord simplement connexe de dimension au moins 4 et que B est une e_n-algèbre, alors l'homologie de factorisation de M à coefficients dans B est donnée, sur $\mathbb{R}$, par un complexe explicite :

$$\int_M B \simeq \mathsf{G}_A^\vee \circ^{\mathbb{L}}_{\mathsf{e}_n} B.$$

Dans le cas où $B = S(\Sigma^{1-n}\mathfrak{g})$ est l'algèbre enveloppante supérieure d'une algèbre de Lie, nous reprenons des arguments de Félix et Thomas [FT04] pour démontrer que $\int_M S(\Sigma^{1-n}\mathfrak{g})$ se calcule avec un complexe de Chevalley–Eilenberg (Proposition 2.5.6), un résultat précédemment obtenu par Knudsen [Knu16] par des méthodes complètement différentes.

Enfin, en utilisant la preuve de Giansiracusa et Salvatore [GS10] de la formalité de l'opérade des petits disques à repère, nous généralisons notre résultat à S^2, la seule surface simplement connexe (Theorem 2.6.6).

Résultats du Chapitre 3

Dans le troisième chapitre, nous étendons les résultats précédents à une large classe de variétés à bord.

Nous nous concentrons sur le cas des variétés admettant un « modèle à dualité de Poincaré–Lefschetz ». Cette notion est une généralisation de « joli modèle surjectif » (*"surjective pretty model"* en anglais), une notion due à Cordova Bulens, Lambrechts et Stanley [CLS15a] (voir aussi [LS05a]). L'intuition derrière cette notion provient de la dualité de Poincaré–Lefschetz. Soit N une variété orientée compacte sans bord de dimension n et $K \subset N$ un sous-polyhèdre, et soit $M = N - \tilde{K}$ est la variété à bord obtenue en retirant un épaississement de K. Alors la cohomologie de M se calcule à partir de la cohomologie de N en « tuant » les classes duales des classes d'homologie provenant de K. Concrètement, un joli modèle surjectif est construit à partir des données suivantes :

- une CDGA à dualité de Poincaré P (correspondant à un modèle de N) ;
- une CDGA connexe Q satisfaisant $Q^{\geq n/2-1} = 0$ (correspondant à un modèle de K) ;
- un morphisme surjectif de CDGAS $\psi : P \to Q$.

La dualité de Poincaré de P induit un isomorphisme $\theta_P : P \to P^\vee[-n]$ entre P et la désuspension de son dual. On définit l'application $\psi^! : Q^\vee[-n] \to P$ comme la composition $\theta_P^{-1} \circ \psi^\vee[-n]$. Notons que $\psi\psi^! = 0$ pour des raisons de degré. Le joli modèle surjectif (correspondant à un modèle de l'inclusion $\partial M \subset M = N - K$) associé à $\psi : P \to Q$ est alors donné par :

$$\lambda = \psi \oplus \mathrm{id} : B = P \oplus_{\psi^!} Q^\vee[1-n] \to B_\partial = Q \oplus Q^\vee[1-n],$$

où $P \oplus_{\psi^!} Q^\vee[1-n]$ est le cône (conoyau homotopique) de $\psi^!$. La CDGA B_∂ est une CDGA à dualité de Poincaré de dimension formelle $n-1$, et B est quasi-isomorphe au quotient $A = P/I$ de la CDGA à dualité de Poincaré P par son idéal $I = \operatorname{im} \psi^!$. La dualité de Poincaré–Lefschetz se traduit en l'existence d'un isomorphisme $A \cong K^\vee[-n]$, où $K = \ker \psi \cong \ker \lambda$ est un modèle des formes relatives sur $(M, \partial M)$.

Un exemple éclairant de joli modèle surjectif est celui de $(M, \partial M) = (D^n, S^{n-1})$. On peut voir $M = D^n$ comme une sphère $N = S^n$ à laquelle on a retiré l'épaississement d'un point $K = \{*\}$. En appliquant le dictionnaire ci-dessus, on prend donc $P = H^*(S^n) = S(\mathrm{vol}_n)/(\mathrm{vol}_n^2)$ comme modèle de N et $Q = H^*(\{*\}) = \mathbb{R}$ comme modèle de K, l'application $\psi : P \to Q$ étant simplement l'augmentation. L'application $\psi^! : Q^\vee[-n] \to P$ est donnée par $\psi^!(1_Q^\vee) = \mathrm{vol}_n$. Le modèle $P \oplus_{\psi^!} Q^\vee[1-n]$ est de dimension 3, engendré par 1_P en degré 0, $1_Q^\vee$ en degré $n-1$ et vol_n en degré n. Tous les produits non-triviaux sont nuls et $d(1_Q^\vee) = \mathrm{vol}_n$.

Nous ne savons pas si toutes les variétés orientées à bord admettent un joli modèle surjectif, contrairement aux variétés simplement connexes (donc orientées) sans bord qui admettent toutes un modèle à dualité de Poincaré. Si M et ∂M sont simplement connexes, nous pouvons, sous l'hypothèse supplémentaire que $\dim M \geq 7$, construire ce que nous appelons un « modèle à dualité de Poincaré–Lefschetz » $\lambda : B \to B_\partial$ de $(M, \partial M)$. La caractéristique principale de ces modèles est l'existence d'un accouplement non-dégénéré entre un quotient $A := B / \ker \theta \simeq B$ et le noyau de λ qui modélise l'accouplement entre $H^*(M)$ et $H^*(M, \partial M)$, comme pour les jolis modèles surjectifs. L'existence d'un modèle à DPL est suffisante pour reprendre toutes les constructions précédentes et obtenir le même modèle $\mathsf{G}_A(k)$ de $\Omega^*_{\mathrm{PA}}(\mathrm{Conf}_k(M))$, ainsi que le modèle $\mathsf{SGraphs}_R^\varphi$ de $\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M)$ (tout en étant compatible avec les structures de comodule si M est parallélisée).

À partir d'un tel joli modèle surjectif, il est possible de reprendre presque mot pour mot la définition de $\mathsf{G}_A(k)$ du Chapitre 2. Si par exemple $M = D^n$, nous obtenons $\mathsf{G}_A \cong \mathsf{e}_n^\vee$ vu comme un comodule à droite sur lui-même. En réutilisant les techniques du Chapitre 2, nous démontrons que $\mathsf{G}_A(k)$ a les mêmes nombres de Betti sur $\mathbb{Q}$ que $\mathrm{Conf}_k(M)$. Nous définissons également une version « perturbée » $\tilde{\mathsf{G}}_A$ de G_A, qui est isomorphe à G_A comme dg-module mais pas comme algèbre, et nous montrons que $\mathsf{G}_A(k)$ est un modèle sur $\mathbb{R}$ de $\mathrm{Conf}_k(M)$.

Il est possible d'adapter la compactification de Fulton–MacPherson FM_M aux variétés compactes à bord, en autorisant des points à devenir infinitésimalement proches les uns des autres, mais aussi en autorisant des points de l'intérieur de la variété à devenir infinitésimalement proches du bord. On obtient ainsi une compactification de l'espace de configuration coloré :

$$\begin{aligned}\mathrm{Conf}_{k,l}(M) &:= \{(x_1, \dots, x_k, y_1, \dots, y_l) \in \mathrm{Conf}_{k+l}(M) \mid x_i \in \partial M, y_j \in \mathring{M}\} \\ &\xrightarrow{\sim} \mathsf{SFM}_M(k,l).\end{aligned}$$

Si M est parallélisée, la collection $\mathsf{SFM}_M = \{\mathsf{SFM}_M(k,l)\}_{k \geq 0, l \geq 0}$ est un module à droite sur une opérade $\mathtt{SFM}_n$ qui est construite de manière similaire à $\mathtt{FM}_n$ et est faiblement équivalent à l'opérade Swiss-Cheese n-dimensionnelle.

Willwacher [Wil15] a construit un modèle en graphes $\mathtt{SGraphs}_n$ de l'opérade $\mathtt{SFM}_n$, similaire au modèle $\mathtt{Graphs}_n$ de Kontsevich pour l'opérade $\mathtt{FM}_n$. Comme dans le Chapitre 2, nous adaptons la construction de Willwacher au « cas étiqueté » pour construire un comodule de Hopf à droite $\mathtt{SGraphs}_R^{c_\varphi, z_\varphi^S}$ sur $\mathtt{SGraphs}_n$, et nous obtenons des quasi-isomorphismes de CDGAs (compatibles avec les structures de comodule le cas échéant) :

$$\mathtt{SGraphs}_R^{\varphi}(k,l) \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathtt{SFM}_M(k,l)),$$

$$\mathtt{G}_A(l) \xleftarrow{\sim} \mathtt{SGraphs}_R^{c_\varphi, z_\varphi^S}(0,l) \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathtt{SFM}_M(0,l)) \simeq \Omega^*_{\mathrm{PA}}(\mathrm{Conf}_l(M)).$$

Contents

0 Introduction and Context

In this thesis, we study the homotopy of configuration spaces of manifolds using ideas coming from the theory of operads. These configuration spaces consist of collections of pairwise distinct points in a given manifold. We address the problems of their homotopy invariance and the definition of rational models for them. We adapt constructions of Kontsevich [Kon99], who proved the formality of (compactified) configuration spaces of Euclidean spaces as an operad.

The notion of an operad was initially introduced for the study of iterated loop spaces in homotopy theory [May72; BV73]. The theory of operads has been considerably renewed since the mid-nineties, when, after insights of Kontsevich [Kon93], Ginzburg and Kapranov [GK94] proved in a seminal work that some duality phenomena in algebra could be interpreted in terms of operads. Since then, a number of new applications of operads have been discovered in several fields of mathematics.

In this introduction, we first explain what operads are in Section 0.1. Then we describe two important families of operads, the little disks operads (Section 0.1.1) and the Swiss-Cheese operads (Section 0.1.2). In Section 0.2, we turn to configuration spaces of manifolds. Section 0.2.1 is devoted to closed manifolds, whereas Section 0.2.2 is about compact manifolds with boundary.

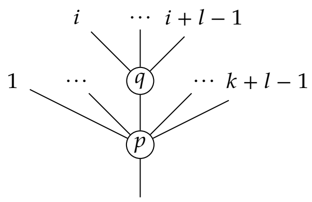
0.1 Operads

Roughly speaking, an operad is an object that governs a category of algebras. The central idea can be understood by analogy with the theory of group representations. In this analogy, the operad corresponds to the group, and the algebras over the operad correspond to the representations of the group.

If a group is given with a presentation by generators and relations, then the associated category of representations can also be defined in terms of the actions of the generators, which have to satisfy the relations. In our analogy, a category of algebras is defined in terms of generating operations and relations. For example, the structure of an associative algebra is defined in terms of a product as a generating operation together with associativity as a relation. The structure of a commutative algebra is defined similarly, with the condition that the product is symmetric, Lie algebras are defined in terms of an antisymmetric Lie bracket and the Jacobi relation, and so on. Each of these definitions can be interpreted as the definition of an operad by generators and relations which governs the given category of algebras.

It is interesting to study the operad itself, independently of any presentation by generators and relations, just like we study groups. This new perspective enables us to consider morphisms, sub-operads, quotients, extensions… Knowledge about an operad then translates into knowledge about algebras over that operad.

Let us explain more precisely what operads are.



The algebraic structures governed by operads are those that can be described in terms of operations with a finite number of inputs and exactly one output. An operad[1] P is a collection $\mathsf{P} = \{\mathsf{P}(k)\}_{k\geq 0}$ of abstract "operations". An element of $\mathsf{P}(k)$ can be viewed as a operation with k inputs ("of arity k") and one output. There is an action of the symmetric group Σ_k on $\mathsf{P}(k)$ which models the permutations of the input of an operation. There are insertion morphisms:

$$\circ_i : \mathsf{P}(k) \otimes \mathsf{P}(l) \to \mathsf{P}(k+l-1), \quad 1 \leq i \leq k,$$

which model composition of operations, similarly to how multiplication in a group G corresponds to composition of operations on representations. Finally, there is an identity $\mathrm{id} \in \mathsf{P}(1)$ which is a neutral element for composition.

To illustrate this definition, one can look at the prototypical example, the *endomorphism operad*. Given some object X in a symmetric monoidal category, the endomorphism operad End_X is defined by $\mathsf{End}_X(n) := \mathrm{Hom}(X^{\otimes n}, X)$. The symmetric group action acts by permuting inputs:

$$(f \cdot \sigma)(x_1, \dots, x_k) = f(x_{\sigma(1)}, \dots, x_{\sigma(k)}),$$

and the insertion operation is given by composition of morphisms:

$$(f \circ_i g)(x_1, \dots, x_{k+l-1}) = f(x_1, \dots, x_{i-1}, g(x_i, \dots, x_{i+l-1}), x_{i+l}, \dots, x_{k+l-1}).$$

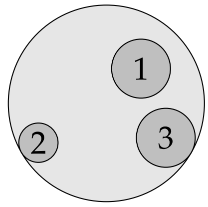
Finally, the identity $\mathrm{id} \in \mathsf{End}_X(1)$ is simply the identity of X. An algebra X over an operad P is by definition a morphism of operads $\mathsf{P} \to \mathsf{End}_X$, just like a group representation is a morphism $G \to \mathrm{End}(V)$. As another example, an operad with only operations of arity 1 is the same thing as a monoid, and an algebra over such an operad is the same thing as a representation of the corresponding monoid. We refer to the books [LV12; Fre17] for a more extensive treatment of operads.

[1]To be precise, by an "operad" we mean a symmetric, one-colored operad. There exists other variants: non-symetric operads, cyclic operads, colored operads, etc.

0.1.1 Little disks operads

There exists a certain family of topological operads of particular interest, the little n-disks operads $\mathtt{D}_n$ (or equivalently, the little n-cubes operads). These are precisely the operads which appeared in the original study of iterated loop spaces.

An element of $\mathtt{D}_n(k)$ is an ordered configuration of k little n-disks with disjoint interiors in the unit disk D^n. Each disk of the configuration represents an embedding of D^n into itself, obtained by the composition of a translation and a positive homothety. The set $\mathtt{D}_n(k)$ is endowed with the compact-open topology of embeddings. The symmetric group action renumbers the disks in a configuration, and insertion is given by composition of embeddings.



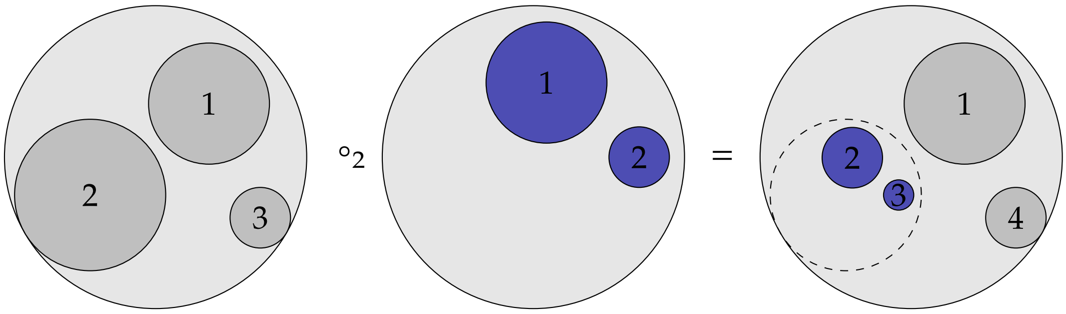


Figure 0.1.1: Composition of embeddings

Almost by definition, an n-fold loop space $\Omega^n X$ is a $\mathtt{D}_n$-algebra. The "recognition principle" [May72; BV73] asserts that the converse is true: under technical conditions, a (grouplike) $\mathtt{D}_n$-algebra is weakly homotopy equivalent to an n-fold loop space.

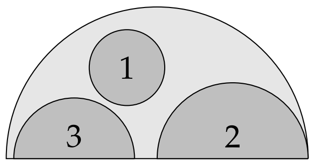
Since this first homotopy-theoretical application, the little disks operads have proved to be useful in a number of settings. Let us mention the Deligne conjecture [KS00; MS02], which asserts that the Hochschild cochains $C^*(A; A)$ of an associative algebra are equipped with an action of $\mathtt{D}_2$; the formality theorem for Hochschild cochains and its application to deformation quantization of Poisson manifolds [Kon99; Tam98; Kon03]; Goodwillie–Weiss calculus and the computation of embedding spaces and long knots [Sin06; LTV10; AT14; DH12; BW13], as well as the "covariant version" of embedding calculus that is factorization homology [BD04; Lur09; Lur16; AF15; CG17].

A fundamental result about the little disks operads is that they are *formal* over $\mathbb{Q}$ [Kon99; Tam03; LV14; FW15]: the operad of chains $C_*(\mathtt{D}_n; \mathbb{Q})$ is quasi-isomorphic to its homology $\mathtt{e}_n := H_*(\mathtt{D}_n; \mathbb{Q})$. Thus it suffices to study the operads

e_n, which have a simple combinatorial description: $\mathsf{e}_1 = \mathsf{Ass}$ governs associative algebras, and for $n \geq 2$, e_n governs $(n-1)$-Poisson algebras [Coh76]. When $n = 2$, the formality of D_2 depends on the choice of a Drinfeld associator, something which gives a universal way to build a braided monoidal category out of some Lie-algebraic data (see e.g. [Fre17, Chapter I.10] for details).

0.1.2 The Swiss-Cheese operads

Colored operads (a.k.a. multicategories), a generalization of operads, are used to describe algebraic structures shaped on multiple objects. In this setting, the inputs and the output of an operation are all labeled by "colors", and composition is only defined if the colors match. In the definition of an algebra over such a colored operad, each color corresponds to a given object.



The Swiss-Cheese operad $\mathsf{SC} = \mathsf{SC}_2$ [Vor99] is a 2-colored operad which governs the action of a D_2-algebra on a D_1-algebra by a central morphism. An operation in SC is given by the embedding of disks and upper half-disks in the unit upper half-disk. It is related to the OCHAs of Kajiura–Stasheff [KS06b; Hoe09] and there is a Swiss-Cheese version of the Deligne conjecture [DTT11]. Higher dimensional variants SC_n govern the action of a D_n-algebra on a D_{n-1}-algebra.

Unlike the little disks operads, the Swiss-Cheese operads are not formal [Liv15] and one cannot recover SC_n from $\mathsf{sc}_n = H_*(\mathsf{SC}_n)$ over $\mathbb{Q}$. Our first goal in this thesis was to find a model for the Swiss-Cheese operad SC_2 that fixes this lack of formality. The homology sc_2 splits as a "Voronov product" $\mathsf{e}_2 \otimes \mathsf{e}_1$ and governs the action of a Gerstenhaber algebra on an associative algebra via a central map. Theorem A describes a model in groupoids for SC_2 which governs the action of a braided monoidal category on a monoidal category via a central monoidal functor. Theorem B describes a model over $\mathbb{Q}$ for SC_2, which uses a Drinfeld associator to construct a kind of "twisted" Voronov product between a rational model for e_2 (built out of chord diagrams) and a model for e_1.

0.2 Configuration spaces

0.2.1 Closed manifolds

Let M be an n-dimensional manifold; its kth configuration space is:

$$\mathrm{Conf}_k(M) = \{x \in M^k \mid \forall 1 \leq i < j \leq k, x_i \neq x_j\}.$$

This construction is obviously not homotopy invariant for open manifolds when $k \geq 2$, as for example $\mathrm{Conf}_2(\mathbb{R}^n) \simeq S^{n-1}$ for $n \in \mathbb{N}$. Even if we restrict our attention to closed manifolds, $\mathrm{Conf}_k(-)$ is still not a homotopy invariant [LS05b]. The counterexample (lens spaces) is not simply connected, and the homotopy invariance for simply connected closed manifolds remains an open question.

Some results are known. The homotopy type of $\Omega\,\mathrm{Conf}_k(M)$ only depends on the homotopy type of $(M, \partial M)$ for connected compact manifolds [Lev95]. Configuration spaces are also known to be a *stable* homotopy invariant [AK04] of closed manifolds. If M is a smooth projective complex manifold, then the rational homotopy type of $\mathrm{Conf}_k(M)$ only depends on the one of M [Kri94]. The same result holds for $k = 2$ when M is a closed manifold and either 2-connected [LS04] or simply connected and even dimensional [Cor15].

We prove, as a corollary of the main result of Chapter 2 (Theorem C), that the real homotopy type of $\mathrm{Conf}_k(M)$ only depends on the real homotopy type of M when the manifold is closed, smooth, simply connected and of dimension at least 4.

When working over $\mathbb{Q}$ (or $\mathbb{R}$) with simply connected manifolds, we can use Sullivan's [Sul77] theory of rational models to study topological spaces up to rational equivalence. Closed, simply connected manifolds are known to have models satisfying a kind of chain-level Poincaré duality [LS08b]. Out of a Poincaré duality model of M, we give an explicit real model for $\mathrm{Conf}_k(M)$. This model had been conjectured by Lambrechts–Stanley [LS08a] in the general case, and had previously been studied in some special cases [CT78; BG91; Kri94; Tot96; FT04; LS04; Cor15].

Our proof relies on Kontsevich's proof of the formality of the little disks operads. The configuration spaces $\mathrm{Conf}_k(\mathbb{R}^n)$ admit compactifications $\mathsf{FM}_n(k)$ due to Fulton–MacPherson [FM94] (and Axelrod–Singer [AS94] in the real case). An element of $\mathsf{FM}_n(k)$ can roughly be seen as a configuration of k points in $\mathbb{R}^n$, where the points are allowed to become "infinitesimally close". One can insert such an infinitesimal configuration into another, and the spaces $\mathsf{FM}_n(-)$ assemble into an operad weakly equivalent to the little disks operads. Kontsevich's proof uses piecewise semi-algebraic (PA) forms and integrals along the fiber to build an explicit equivalence between $C_*(\mathsf{FM}_n)$ and e_n.

Similarly, if M is a closed manifold, then $\mathrm{Conf}_k(M)$ can be compactified into $\mathsf{FM}_M(k)$. If M happens to be framed, then the spaces $\mathsf{FM}_M(-)$ assemble to form a right module over the operad FM_n. This extra structure is what allows us to prove Theorem C. When M is framed, we then obtain that the action of the little disks operad on the configuration spaces is compatible with our model. We are able, using this additional result and a comparison statement between the Cohen–Taylor spectral sequence [CT78] and the Bendersky–Gitler spectral sequence [BG91] established by Félix–Thomas [FT04], to recover a theorem of Knudsen [Knu16] about the factorization homology of a framed manifold with coefficients in the higher enveloping algebra of a Lie algebra.

0.2.2 Compact manifolds with boundary

In the last chapter, we study configuration spaces of compact manifolds with boundary. As for closed manifolds, homotopy invariance of configuration spaces of compact manifolds with boundary is an open question. It is known that if the manifold and its boundary are both 2-connected, then the rational homotopy type of $\mathrm{Conf}_2(M)$ only depends on the rational homotopy type of the pair $(M, \partial M)$ [CLS15b].

The manifolds we consider are those which admit a "Poincaré–Lefschetz duality model" These are a generalization of surjective pretty models as defined in [CLS15a]. Briefly, if N is a closed manifold and $K \subset N$ is a sub-polyhedron, to obtain a model for $M = N - K$ one takes a Poincaré duality model for N and then "kills" (using a mapping cone) the cohomology classes coming from K. We can apply these ideas to manifolds with boundary which are obtained by removing an open neighborhood of a sub-polyhedron from a closed manifold. Manifolds admitting a surjective pretty model include closed manifolds, 2-connected manifolds with 2-connected boundary satisfying an algebraic retraction property, disk bundles of even rank of simply connected closed manifolds, and spaces obtained by removing a high-codimensional sub-polyhedron from a 2-connected manifold [CLS15a; CLS15b] (see Theorem 3.1.12 for details). More generally, if M is simply connected with simply connected boundary, we can prove that it admits a Poincaré–Lefschetz duality model as soon as $n \geq 7$, and we recover the same results.

We first use a kind of chain-level Poincaré–Lefschetz duality and cubical diagrams to compute the Betti numbers of $\mathrm{Conf}_k(M)$, as was done for closed manifolds in [LS08a]. We then reuse the methods of Chapter 2 to provide an explicit real model for $\mathrm{Conf}_k(M)$. One must adapt the Fulton–MacPherson compactifications to deal with the boundary of M, and we obtain as Theorem D that the model conjectured in [CLS15b] is a real model for $\mathrm{Conf}_k(M)$. This model only depends on a model of M, therefore, as a corollary, we obtain the real homotopy invariance of $\mathrm{Conf}_k(-)$ for smooth simply connected manifolds with simply connected boundary of dimension ≥ 5 (either admitting a pretty model or of dimension ≥ 7).

The compactified configuration spaces on a framed manifold with boundary inherit an action of an operad SFM_n, weakly equivalent to the Swiss-Cheese operad SC_n. In Theorem F, we describe a model for the resulting module over SFM_n, using a graphical model of SC_n found by Willwacher [Wil15].

1 Swiss-Cheese Operad and Drinfeld Center

The little disks operads D_n of Boardman–Vogt and May [BV73; May72] govern algebras which are associative and (for $n \geq 2$) commutative up to homotopy. For $n = 2$, one can see that the fundamental groupoid of D_2 forms an operad $\pi\mathsf{D}_2$ equivalent to an operad in groupoids PaB, called the operad of parenthesized braids, which governs braided monoidal categories [Fre17, §I.6]. Since the homotopy of D_2 is concentrated in degrees ≤ 1, this is enough to recover D_2 up to homotopy. For $n = 1$, one can also easily see that $\pi\mathsf{D}_1$ is equivalent to an operad PaP, called the operad of parenthesized permutations, which governs monoidal categories.

The Swiss-Cheese operad $\mathsf{SC} = \mathsf{SC}_2$ of Voronov [Vor99] governs the action of a D_2-algebra on a D_1-algebra by a central morphism. As explained by Hoefel [Hoe09], the Swiss-Cheese operad is intimately related to the "Open-Closed Homotopy Algebras" (OCHAs) of Kajiura and Stasheff [KS06a], which are of great interest in string field theory and deformation quantization.

We aim to study the fundamental groupoid of SC, which is still an operad. This fundamental groupoid is again enough to recover SC up to homotopy. In a first step, we established the following theorem:

Theorem A (See Theorem 1.2.11 and Corollary 1.3.4)**.** *The fundamental groupoid operad $\pi\mathsf{SC}$ is equivalent to an operad PaPB whose algebras (in the category of categories) are triples $(\mathsf{M}, \mathsf{N}, F)$, where N is a monoidal category, M is a braided monoidal category, and $F : \mathsf{M} \to \mathcal{Z}(\mathsf{N})$ is a strong braided monoidal functor from M to the Drinfeld center $\mathcal{Z}(\mathsf{N})$ of N.*

In this theorem, the monoidal categories have no unit. We also consider the unitary version SC_+ of the Swiss-Cheese operad, and we obtain (Proposition 1.3.9) an extension PaPB_+ of the model where the monoidal categories have a strict unit and the functor strictly preserves the unit.

The result of Theorem A is a counterpart for operads in groupoids of statements of [Gin15, Proposition 31] and [AFT17, Example 2.13] about the ∞-category of factorization algebras on the upper half plane.

Based on "Swiss-Cheese Operad and Drinfeld Center", *Israel J. Math.* (to appear), arXiv: 1507.06844.

In a second step, we rely on the result of Theorem A to construct an operad rationally equivalent (in the sense of rational homotopy theory) to the completion of the Swiss-Cheese operad. To this end, we use Drinfeld associators, which we see as morphisms $\mathsf{PaB}_+ \to \widehat{\mathsf{CD}}_+$, where $\widehat{\mathsf{CD}}_+$ is the completed operad of chord diagrams. The existence of such a rational Drinfeld associator is equivalent to the rational formality of D_2, but the inclusion $\mathsf{D}_1 \to \mathsf{D}_2$ is not formal; equivalently, the constant morphism $\mathsf{PaP}_+ \to \widehat{\mathsf{CD}}_+$ does not factor through a Drinfeld associator. We prove the following theorem:

Theorem B (See Theorem 1.4.21)**.** *Given a choice of Drinfeld associator ϕ, there is an operad in groupoids $\mathsf{PaP}\widehat{\mathsf{CD}}_+^{\phi}$ built using chord diagrams, parenthesized permutations, and parenthesized shuffles, which is rationally equivalent to $\pi\mathsf{SC}_+$.*

The Swiss-Cheese operad is not formal [Liv15], thus it cannot be recovered from its homology $H_*(\mathsf{SC})$. We use the splitting of $H_*(\mathsf{SC})$ as a product [Vor99] to build an operad in groupoids $\widehat{\mathsf{CD}} \times_+ \mathsf{PaP}$, and we compare it to our rational model of SC.

Independently of the author, Willwacher [Wil15] found a different model for the Swiss-Cheese operad in any dimension $n \geq 2$ that uses graph complexes. His model extends Kontsevich's [Kon99] quasi-isomorphism $\mathsf{Graphs}_n \xrightarrow{\sim} \Omega^*(\mathsf{D}_n)$ from the proof of the formality of D_n, whereas our model extends (after passing to classifying spaces) Tamarkin's [Tam03] model $\mathrm{B}\,\hat{\mathrm{U}}\mathfrak{p} = \mathrm{B}\,\widehat{\mathsf{CD}}$ of D_2. Thus, in contrast to Willwacher's model, our own model is related to Drinfeld's original approach to quantization. It would be interesting to compare the two, as was done by Ševera and Willwacher [ŠW11] for the little 2-disks operad.

This paper is organized as follows: in Section 1.1, we recall some background on the Swiss-Cheese operad and relative operads; in Section 1.2, we construct two algebraic models for the Swiss-Cheese operad; in Section 1.3, we describe what the algebras over these models are, using Drinfeld centers; and in Section 1.4, we construct a rational model in groupoids for the Swiss-Cheese operad using chords diagrams and Drinfeld associators.

1.1 Background

The little n-disks operad D_n is built out of configurations of embeddings of little n-disks (whose images have disjoint interiors) in the unit n-disk, and operadic composition is given by composition of such embeddings – see [BV73; May72] for precise definitions.

The fundamental groupoid $\pi\mathsf{D}_2$ of the little disks operads is weakly equivalent to an operad in groupoids called CoB, the operad of colored braids [Fre17, §I.5], which we now recall. Given an integer $r \geq 0$, the object set of $\mathsf{CoB}(r)$ is the symmetric group Σ_r. For $\sigma, \sigma' \in \Sigma_r$, the morphisms in $\mathrm{Hom}_{\mathsf{CoB}(r)}(\sigma, \sigma')$ are given by isotopy classes of braids with r strands which respect the permutations, i.e. if we label the beginning of each strand as $\sigma(1), \sigma(2), \dots, \sigma(r)$ and the end of each strand with $\sigma'(1), \sigma'(2), \dots, \sigma'(r)$, then the labelings match (see Figure 1.1.1 for an example). This way, given a source permutation and a braid, the target permutation is fixed and we will not draw it on pictures.

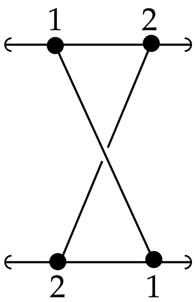


Figure 1.1.1: Example of element in $\mathsf{CoB}(2)$

We will also consider colored operads, which are also called multicategories. These are used to study several algebraic structures at once, together with possible relations between them. Compared with standard operads, a colored operad has an extra data, the set of its "colors", and for a given operation, each input and the output is assigned a color. There is an identity for each color, and composition is only possible if the input and output match. An "algebra" over a colored operad is in fact given by several objects, one for each color, and the operations act on them according to how their inputs and outputs are colored. For example, given an operad P, there is a canonical operad $\vec{\mathsf{P}}$ with two colors, whose algebras are triplets (A, B, f) where A and B are P-algebras and $f : A \to B$ is a morphism of P-algebras.

The Swiss-Cheese operad SC is an operad with two colors, $\mathfrak{c}$ and $\mathfrak{o}$ (standing for "closed" and "open"). The space of operations $\mathsf{SC}(x_1, \dots, x_n; \mathfrak{c})$ with a closed output is equal to $\mathsf{D}_2(n)$ if $x_1 = \cdots = x_n = \mathfrak{c}$, and it is empty otherwise. The space $\mathsf{SC}(x_1, \dots, x_n; \mathfrak{o})$ is the space of configurations of embeddings of full disks (corresponding to the color $\mathfrak{c}$) and half disks (corresponding to the color $\mathfrak{o}$), with disjoint interiors, inside the unit upper half disk (see Figure 1.1.2 for an example). Composition is again given by composition of embeddings.

The Swiss-Cheese operad is an example of a relative operad [Vor99]: it can be seen as an operad in the category of right modules (in the sense of [Fre09]) over another operad.

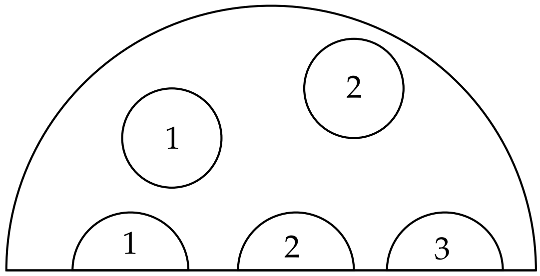


Figure 1.1.2: Example of an element in $\mathtt{SC}(3,2)$

Definition 1.1.1. Let $\mathtt{P}$ be a (symmetric, one-colored) operad. A **relative operad over** $\mathtt{P}$ is an operad $\mathtt{Q}$ in the category of right modules over $\mathtt{P}$. Equivalently, it is a two-colored operad $\mathtt{Q}$ (where the two colors are called $\mathfrak{c}$ and $\mathfrak{o}$) such that:

$$\mathtt{Q}(x_1,\dots,x_n;\mathfrak{c}) = \begin{cases} \mathtt{P}(n), & \text{if } x_1 = \dots = x_n = \mathfrak{c}; \\ \emptyset, & \text{otherwise.} \end{cases}$$

$$\mathtt{Q}(n,m) := \mathtt{Q}(\underbrace{\mathfrak{o},\dots,\mathfrak{o}}_{n},\underbrace{\mathfrak{c},\dots,\mathfrak{c}}_{m};\mathfrak{o}) = (\mathtt{Q}(n))(m).$$

If $\mathtt{Q}$ is such a relative operad, we will write $\mathtt{Q}^{\mathfrak{c}}(m) := \mathtt{P}(m)$.

The Swiss-Cheese operad $\mathtt{SC}$ is a relative operad over the little disks operad $\mathtt{D}_2$. We also consider the unitary version of the Swiss-Cheese operad $\mathtt{SC}_+$, which is a relative operad over the unitary version of the little disks operad $\mathtt{D}_2^+$, and which satisfies $\mathtt{SC}_+(0,0) = *$. Composition with the nullary elements simply forgets half disks or full disks of the configuration.

Remark 1.1.2. We consider a variation of the Swiss-Cheese operad, where we allow operations with only closed inputs and an open output, whereas in Voronov's definition these configurations are forbidden. We write $\mathtt{SC}^{\mathrm{vor}}$ for Voronov's version, so that $\mathtt{SC}^{\mathrm{vor}}(0,m) = \emptyset$ while $\mathtt{SC}(0,m) \simeq \mathtt{D}_2(m) \neq \emptyset$.

1.2 Permutations and braids

1.2.1 Colored version

We first define an operad in groupoids $\mathtt{CoPB}$, the **operad of colored permutations and braids**. It is an operad relative over $\mathtt{CoB}$, the operad of colored braids [Fre17, §I.5].

Let $D^+ = \{z \in \mathbb{C} \mid \Im z \geq 0, |z| \leq 1\}$ be the upper half disk, and let

$$\mathrm{Conf}(n,m) = \{(z_1,\dots,z_n,u_1,\dots,u_m) \in D^+ \mid \Im z_i = 0, \Im u_j > 0, z_i \neq z_j, u_i \neq u_j\}$$

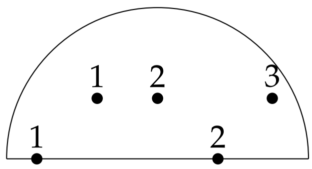
be the set of configurations of n points on the real interval $[-1, 1]$ and m points in the upper half disk.

The disk-center mapping $\omega : \mathsf{SC}(n, m) \xrightarrow{\sim} \mathrm{Conf}(n, m)$, sending each disk to its center, is a weak equivalence [Vor99]. Let Σ_k be the kth symmetric group, and let $\mathrm{Sh}_{n,m}$ be the set of (n, m)-shuffles:

$$\mathrm{Sh}_{n,m} = \{\mu \in \Sigma_{n+m} \mid \mu(1) < \cdots < \mu(n), \mu(n+1) < \cdots < \mu(n+m)\}.$$

For every $\mu \in \mathrm{Sh}_{n,m}$, we choose a configuration $c^0_\mu \in \mathrm{Conf}(n, m)$ with n "terrestrial" points (on the real axis) and m "aerial" points (with positive imaginary part), in the left-to-right order given by the (n, m)-shuffle μ. For example we can choose:

$$\mu = (14|235) \in \mathrm{Sh}_{2,3} \rightsquigarrow c^0_\mu =$$



$$\in \mathrm{Conf}(2, 3), \qquad (1.2.1)$$

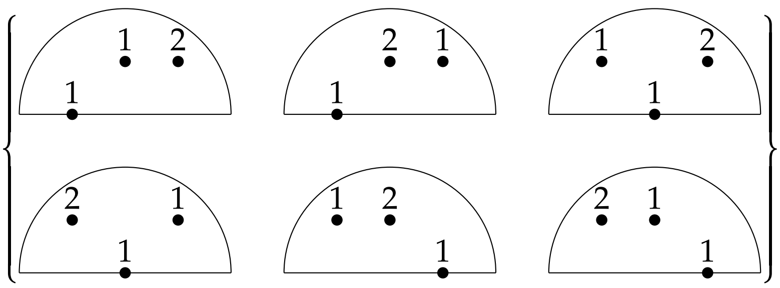
We consider the set:

$$C^0(n, m) = \{\sigma \cdot c^0_\mu\}_{\sigma \in \Sigma_n \times \Sigma_m,\, \mu \in \mathrm{Sh}_{n,m}} \subset \mathrm{Conf}(n, m),$$

where $\Sigma_n \times \Sigma_m$ acts by permuting labels.

Example 1.2.2. For example, $C^0(2, 1)$ can be chosen to be:

$$C^0(2, 1) =$$

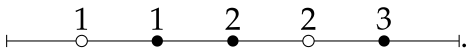


The precise position of the points does not matter for our purposes, only their left-to-right order.

Definition 1.2.3. The groupoid $\mathsf{CoPB}(n, m)$ is the restriction of the fundamental groupoid of $\mathrm{Conf}(n, m)$ to the set $C^0(n, m) \subset \mathrm{Conf}(n, m)$ (i.e. it is its full subcategory with these objects):

$$\mathsf{CoPB}(n, m) := \pi\,\mathrm{Conf}(n, m)|_{C^0(n,m)}.$$

The set $\operatorname{ob} \mathtt{CoPB}(n,m) = C^0(n,m)$ is isomorphic to $\mathrm{Sh}_{n,m} \times \Sigma_n \times \Sigma_m$. We represent these objects by sequences of n "terrestrial" points (drawn in white and labeled by $\{1,\dots,n\}$) and m "aerial" points (drawn in black and labeled by $\{1,\dots,m\}$) on the interval $I = [-1,1]$; the order in which terrestrial and aerial points appear is given by the shuffle. For example, the element in Equation (1.2.1) is represented by:



Morphisms between two such configurations are given by isotopy classes of bicolored braids, where strands between terrestrial points never go behind any other strand, including other terrestrial strands (indeed, they represent paths in the interval $[-1,1]$, and points cannot move over one another in $\mathrm{Conf}_n([-1,1])$, nor can they go behind the paths in the open upper half disk). See Figure 1.2.1 for an example of an element in $\mathtt{CoPB}(2,3)$, and Figure 1.2.2 for the corresponding path in $\mathrm{Conf}(2,3)$.

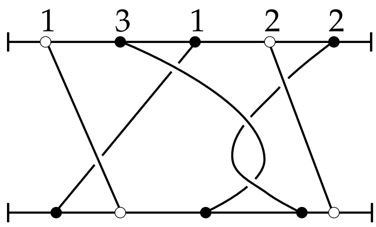


Figure 1.2.1: Element in $\mathtt{CoPB}(2,3)$

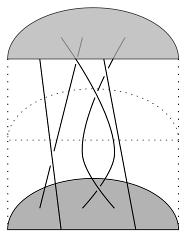
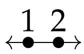


Figure 1.2.2: Corresponding path in $\mathrm{Conf}(2,3)$

To not confuse objects of $\mathtt{CoPB}$ and objects of $\mathtt{CoB}$, and to be coherent with the graphical representation of $\Omega\Omega$ in Section 1.2.2, we draw the objects of $\mathtt{CoB}$ with ends in the shape of chevrons:



$\in \operatorname{ob} \mathtt{CoB}(2)$.

The symmetric sequence $\mathtt{CoPB}(n) = \{\mathtt{CoPB}(n,m)\}_{m \geq 0}$ is a right module over $\mathtt{CoB}$ by inserting a colored braid in a tubular neighborhood of an aerial strand (Figure 1.2.3). Similarly, the operad structure inserts a colored braid in a tubular neighborhood of a terrestrial strand (Figure 1.2.4). One can easily check that this gives a relative operad over $\mathtt{CoB}$ (in the same manner that one checks that $\mathtt{CoB}$ itself is an operad, cf. [Fre17, §I.5]).

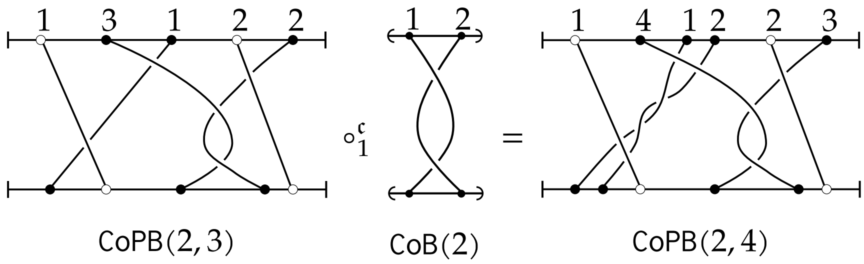


Figure 1.2.3: Definition of the right CoB-module structure

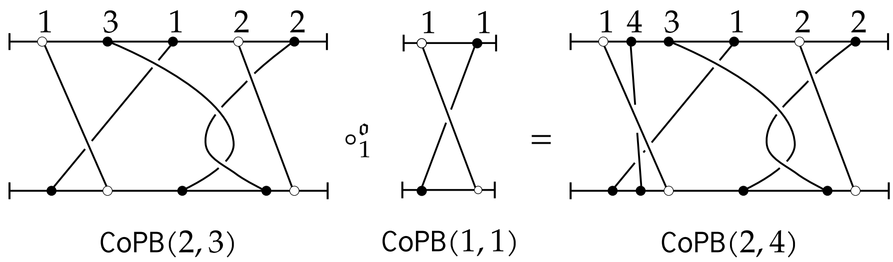


Figure 1.2.4: Definition of the operad structure

1.2.2 Magmas

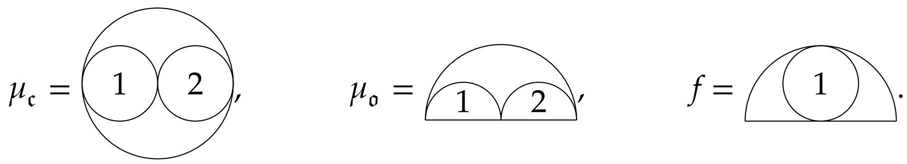
Definition 1.2.4. Let Ω be the **magma operad** (as in [Fre17]), defined as the free symmetric operad $\mathbb{O}(\mu_\mathfrak{c})$ on a single generator $\mu_\mathfrak{c} = \mu_\mathfrak{c}(x_1, x_2)$ of arity 2 (where Σ_2 acts freely on $\mu_\mathfrak{c}$). Its algebras are sets endowed with a product satisfying no further conditions.

Elements of $\Omega(n)$ are parenthesizations of a permutation of n elements, for example $(((x_1x_3)(x_2x_4))x_5) \in \Omega(5)$. The index $\mathfrak{c}$ of $\mu_\mathfrak{c}$ is there to be coherent with the following definition:

Definition 1.2.5. Let $\Omega\Omega = \mathbb{O}(\mu_\mathfrak{c}, f, \mu_\mathfrak{o})$ be the free colored operad on the three generators $\mu_\mathfrak{c} \in \Omega\Omega(\mathfrak{c}, \mathfrak{c}; \mathfrak{c})$, $f \in \Omega\Omega(\mathfrak{c}; \mathfrak{o})$ et $\mu_\mathfrak{o} \in \Omega\Omega(\mathfrak{o}, \mathfrak{o}; \mathfrak{o})$. It is a relative operad over Ω.

An algebra over $\Omega\Omega$ is the data of two magmas M, N, and of a mere function $f : M \to N$ (not necessarily preserving the product).

Lemma 1.2.6. *The suboperad of* $\mathtt{SC}$ *generated by the following three elements is free on those generators:*



Proof. We would like to show that the induced morphism $i : \Omega\Omega \to \mathtt{SC}$, sending the three generators of $\Omega\Omega$ to the elements depicted in the lemma, is an embedding, i.e. an isomorphism onto its image. The image of this induced morphism is by definition the suboperad generated by the three elements, hence the lemma.

The fact that the suboperad of $\mathtt{D}_2$ generated by $\mu_\mathfrak{c}$ is free is given by [Fre17, Proposition I.6.2.2(a)]. Let $\alpha \in \mathtt{SC}(n, m)$ be a configuration, as in Figure 1.2.5. We will build an element of $\Omega\Omega$ which is sent to α under i. This set-level retraction is not necessarily a morphism of operads but still shows that i is injective, which will prove the lemma.

We first regroup the m full disks into connected components $C_1, \dots, C_r$. For each C_i, we consider the center of the middle horizontal interval (in blue on the figure), which we project onto the real line (in red). These points, together with the centers of the n half disks, make up a dyadic configuration on the horizontal diameter of the ambient half disk. By [Fre17, Proposition I.6.2.2(a)], such a dyadic configuration is equivalent to an element $u \in \Omega(n + r)$ (which we see as an iterate of $\mu_\mathfrak{c}$).

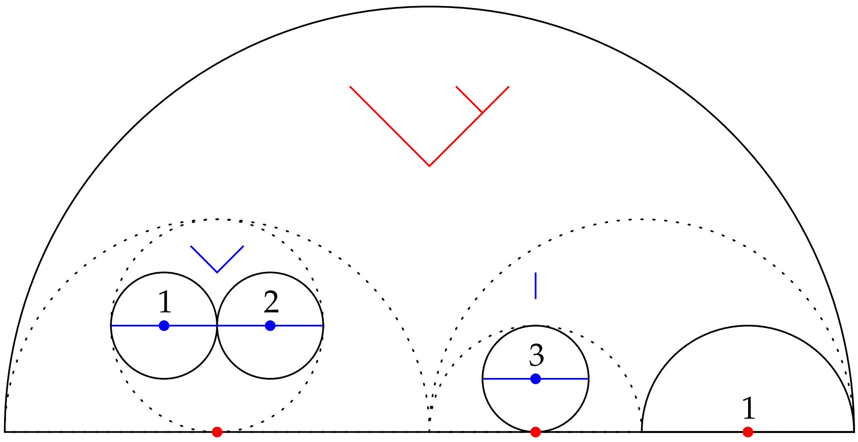
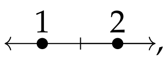
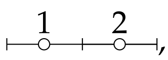


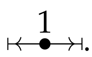
Figure 1.2.5: Example element of $\Omega\Omega(1,3)$

For each C_i (corresponding to an input of u), we apply the same proposition [Fre17, Proposition I.6.2.2(a)] to get an element $v_i \in \Omega(k_i)$ (which we see as a iterate of μ_o), and $\sum k_i = m$. If we plug $f(v_i)$ in the corresponding inputs of u, we get an element of $\Omega\Omega(n,m)$, and by construction this elements gets sent to α by i. $\square$

We consider the following graphical representation for elements of $\Omega\Omega$ with open output, where the generators are represented as follows:

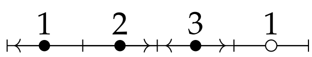
$\mu_c \rightsquigarrow$ 

, $\mu_o \rightsquigarrow$ 

, $f \rightsquigarrow$ 

.

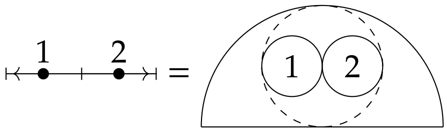
For example, this is the representation of the element of Figure 1.2.5:

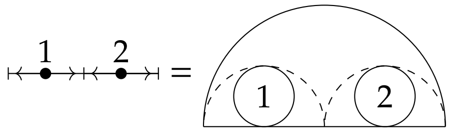
$\mu_o(f(\mu_c(x_1,x_2)),\mu_o(f(x_3),y_1)) \rightsquigarrow$ 

$\in \Omega\Omega(1,3)$

Each μ_o is represented by cutting in half the interval; f is represented by parentheses, and μ_c is again represented by cutting the interval inside the parentheses in half. Closed inputs are represented by black points, while open inputs are represented by white points.

Remark 1.2.7. The parentheses separating the aerial points are really necessary in the representation. For example these are two different objects:

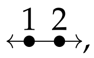
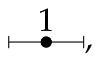
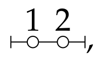
$f(\mu_c(x_1, x_2)) =$ 

$\mu_o(f(x_1), f(x_2)) =$ 

1.2.3 Parenthesized version

We now define PaPB, the **operad of parenthesized permutations and braids**, a relative operad over PaB. The definition of PaPB is given as a pullback of CoPB, similarly to how PaB is a pullback of CoB.

Definition 1.2.8. We consider the morphism $\omega : \Omega\Omega \to \mathrm{ob}\,\mathsf{CoPB}$, given on generators by:

$\mu_c \mapsto$ , $\quad f \mapsto$ , $\quad \mu_o \mapsto$ ,

and we define $\mathsf{PaPB} := \omega^*\mathsf{CoPB}$, the pullback of CoPB along ω. It is an operad in groupoids such that $\mathrm{ob}\,\mathsf{PaPB} = \Omega\Omega$ and

$$\mathrm{Hom}_{\mathsf{PaPB}(n,m)}(u, v) := \mathrm{Hom}_{\mathsf{CoPB}(n,m)}(\omega(u), \omega(v))$$

for $u, v \in \Omega\Omega(n, m)$.

Definition 1.2.9. A **categorical equivalence** is a morphism of operads in groupoids which is an equivalence of categories in each arity. Two operads P and Q are said to be **categorically equivalent** (and we write $\mathsf{P} \simeq \mathsf{Q}$) if they can be connected by a zigzag of categorical equivalences:

$$\mathsf{P} \xleftarrow{\sim} \cdot \xrightarrow{\sim} \ldots \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathsf{Q}.$$

Recall that the fundamental groupoid functor $\pi : (\mathsf{Top}, \times) \to (\mathsf{Gpd}, \times)$ is monoidal, thus the fundamental groupoid of a topological operad is an operad in groupoids.

Remark 1.2.10. Since each arity of the operad SC has homotopy concentrated in degrees ≤ 1, it follows that its fundamental groupoid is enough to recover the homotopy type of the operad through the classifying space construction: $\mathsf{SC} \xrightarrow{\sim} \mathrm{B}\,\pi\mathsf{SC}$.

Theorem 1.2.11. *The operad* PaPB *is isomorphic to the fundamental groupoid of* SC *restricted to the image of* $\Omega\Omega \hookrightarrow \mathsf{SC}$*, and we get a zigzag of categorical equivalences:*

$$\pi\mathsf{SC} \xleftarrow{\sim} \pi\mathsf{SC}|_{\Omega\Omega} \cong \mathsf{PaPB} \xrightarrow{\sim} \mathsf{CoPB}.$$

Proof. The proof of the first part of the proposition is a direct adaptation of the proof of [Fre17, Proposition I.6.2.2(b)]. We note that $\Omega\Omega \subset \operatorname{ob} \pi\mathsf{SC}$ is a suboperad, thus $\pi\mathsf{SC}|_{\Omega\Omega}$ is also a suboperad of $\pi\mathsf{SC}$. For the second part, we note that $\Omega\Omega(n,m)$ meets all the connected components of $\mathsf{SC}(n,m) \sim \Sigma_n \times \mathsf{D}_2(m)$, so the first inclusion is a categorical equivalence. Since $\omega : \Omega\Omega \to \operatorname{ob} \mathsf{CoPB}$ is surjective, the second morphism is also a categorical equivalence. $\square$

1.3 Drinfeld center

1.3.1 Algebras over PaPB

Definition 1.3.1. Let C be a (non-unitary) monoidal category. Its suspension $\Sigma\mathsf{C}$ is a bicategory with a single object. The **Drinfeld center** [Maj91; JS91] of C is the braided monoidal category $\mathcal{Z}(\mathsf{C}) = \operatorname{End}(\mathrm{id}_{\Sigma\mathsf{C}})$. Explicitly, it is given as follows (see also [nLa16] for the more abstract point of view):

- Objects are pairs (X, Ψ), where X is an object of C and $\Psi : (X \otimes -) \to (- \otimes X)$ is a **half-braiding**, i.e. a natural isomorphism such that for all $Y, Z \in \mathsf{C}$ the following diagram commutes:

$$\begin{tikzcd} & X \otimes (Y \otimes Z) \arrow[r, "\Psi_{Y\otimes Z}"] & (Y \otimes Z) \otimes X \arrow[dr, "\alpha"] & \\ (X \otimes Y) \otimes Z \arrow[ur, "\alpha"] \arrow[dr, "\Psi_Y \otimes 1"'] & & & Y \otimes (Z \otimes X) \\ & (Y \otimes X) \otimes Z \arrow[r, "\alpha"'] & Y \otimes (X \otimes Z) \arrow[ur, "1\otimes\Psi_Z"'] & \end{tikzcd}$$

- Morphisms between (X, Ψ) and (Y, Ψ') are morphisms $f : X \to Y$ of C such that, for all $Z \in \mathsf{C}$, the following diagram commutes

$$\begin{tikzcd} X \otimes Z \arrow[r, "f\otimes 1"] \arrow[d, "\Psi_Z"'] & Y \otimes Z \arrow[d, "\Psi'_Z"] \\ Z \otimes X \arrow[r, "1\otimes f"'] & Z \otimes Y \end{tikzcd}$$

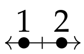
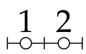
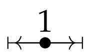
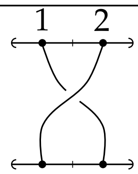
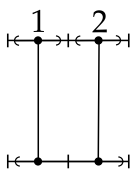
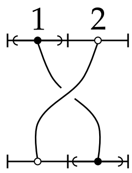
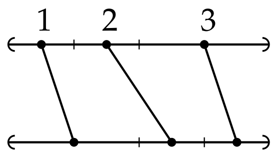
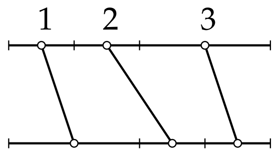
- The tensor product of two objects $(X, \Psi) \otimes (X', \Psi')$ is given by $(X \otimes X', \Psi'')$, where Ψ''_Z is defined by the following diagram (that can be rearranged as an hexagon by inverting both vertical α's, see Figure 1.3.6):

$$\begin{tikzcd}
(X \otimes X') \otimes Y \arrow[rrr, "\Psi''_Y"] \arrow[d, "\alpha"'] & & & Y \otimes (X \otimes X') \\
X \otimes (X' \otimes Y) \arrow[r, "1 \otimes \Psi'_Y"'] & X \otimes (Y \otimes X') \arrow[r, "\alpha^{-1}"'] & (X \otimes Y) \otimes X' \arrow[r, "\Psi_Y \otimes 1"'] & (Y \otimes X) \otimes X' \arrow[u, "\alpha"']
\end{tikzcd}$$

- The braiding $(X, \Psi) \otimes (X', \Psi') \to (X', \Psi') \otimes (X, \Psi)$ is given by $\Psi_{X'}$ and the associator is given by the associator of C.

Example 1.3.2. Let H be a Hopf algebra and consider the category $\mathrm{Rep}(H)$ of its representations. Then the Drinfeld center $\mathcal{Z}(\mathrm{Rep}(H))$ is equivalent to the category $\mathrm{Rep}(D(H))$ of representations of the "Drinfeld double" $D(H)$ of H, which is roughly speaking obtained as a semi-direct product of H with its dual $D(H) \approx H \rtimes H^\vee$ [Dri87].

We consider the following elements of PaPB:

$\mu_\mathfrak{c} \in \operatorname{ob} \mathsf{PaB}(2)$	$\mu_\mathfrak{o} \in \operatorname{ob} \mathsf{PaPB}(2,0)$	$f \in \operatorname{ob} \mathsf{PaPB}(0,1)$	$\tau \in \mathsf{PaB}(2)$
			
$p \in \mathsf{PaPB}(0,2)$	$\psi \in \mathsf{PaPB}(1,1)$	$\alpha_\mathfrak{c} \in \mathsf{PaB}(3)$	$\alpha_\mathfrak{o} \in \mathsf{PaPB}(3,0)$
			

Theorem 1.3.3. *Let* P *be a* $\{\mathfrak{c}, \mathfrak{o}\}$*-colored operad*[1] *in the category of categories, let* $m_\mathfrak{c} \in \operatorname{ob} \mathsf{P}^\mathfrak{c}(2)$, $m_\mathfrak{o} \in \operatorname{ob} \mathsf{P}(2,0)$, $F \in \operatorname{ob} \mathsf{P}(0,1)$ *be objects, and let*

$$\begin{aligned}
a_\mathfrak{c} &: m_\mathfrak{c}(m_\mathfrak{c}(x_1, x_2), x_3) \to m_\mathfrak{c}(x_1, m_\mathfrak{c}(x_2, x_3)), & \pi &: m_\mathfrak{o}(f(x_1), f(x_2)) \to f(m_\mathfrak{c}(x_1, x_2)), \\
t &: m_\mathfrak{c}(x_1, x_2) \to m_\mathfrak{c}(x_2, x_1), & \Psi &: m_\mathfrak{o}(f(x_1), y_1) \to m_\mathfrak{o}(y_1, f(x_1)), \\
a_\mathfrak{o} &: m_\mathfrak{o}(m_\mathfrak{o}(y_1, y_2), y_3) \to m_\mathfrak{o}(y_1, m_\mathfrak{o}(y_2, y_3)),
\end{aligned}$$

[1] The operad will not necessarily be a relative operad, but we will still use the notation $\mathsf{P}(n, m) = \mathsf{P}(\mathfrak{c}^m, \mathfrak{o}^n; \mathfrak{o})$ and $\mathsf{P}^\mathfrak{c}(m) = \mathsf{P}(\mathfrak{c}^m; \mathfrak{c})$.

be isomorphisms. Then there exists a morphism $\theta : \mathsf{PaPB} \to \mathsf{P}$ *such that*

$$\begin{aligned} &\theta(\mu_{\mathfrak{c}}) = m_{\mathfrak{c}}, && \theta(\mu_{\mathfrak{o}}) = m_{\mathfrak{o}}, && \theta(f) = F, && \theta(\alpha_{\mathfrak{c}}) = a_{\mathfrak{c}}, \\ &\theta(\alpha_{\mathfrak{o}}) = a_{\mathfrak{o}}, && \theta(\tau) = t, && \theta(p) = \pi, && \theta(\psi) = \Psi, \end{aligned}$$

(in which case this morphism is unique) if, and only if, the coherence diagrams of Figures 1.3.1 to 1.3.6 commute.

$$\begin{tikzcd}[column sep=tiny]
& m(m(m(x_1,x_2),x_3),x_4) \arrow[dl, "m(a,\mathrm{id})"'] \arrow[dr, "a"] & \\
m(m(x_1,m(x_2,x_3)),x_4) \arrow[d, "a"'] & & m(m(x_1,x_2),m(x_3,x_4)) \arrow[ddl, "a"] \\
m(x_1,m(m(x_2,x_3),x_4)) \arrow[dr, "m(\mathrm{id},a)"'] & & \\
& m(x_1,m(x_2,m(x_3,x_4))) &
\end{tikzcd}$$

Figure 1.3.1: Pentagon for $(m,a) = (m_{\mathfrak{c}}, a_{\mathfrak{c}})$ and $(m_{\mathfrak{o}}, a_{\mathfrak{o}})$

$$\begin{tikzcd}[column sep=tiny]
& m_{\mathfrak{c}}(m_{\mathfrak{c}}(x_1,x_2),x_3) \arrow[dl, "m_{\mathfrak{c}}(t,\mathrm{id})"'] \arrow[dr, "a_{\mathfrak{c}}"] & \\
m_{\mathfrak{c}}(m_{\mathfrak{c}}(x_2,x_1),x_3) \arrow[d, "a_{\mathfrak{c}}"'] & & m_{\mathfrak{c}}(x_1,m_{\mathfrak{c}}(x_2,x_3)) \arrow[d, "t"] \\
m_{\mathfrak{c}}(x_2,m_{\mathfrak{c}}(x_1,x_3)) \arrow[dr, "m_{\mathfrak{c}}(\mathrm{id},t)"'] & & m_{\mathfrak{c}}(m_{\mathfrak{c}}(x_2,x_3),x_1) \arrow[dl, "a_{\mathfrak{c}}"] \\
& m_{\mathfrak{c}}(x_2,m_{\mathfrak{c}}(x_3,x_1)) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=tiny]
& m_{\mathfrak{c}}(x_1,m_{\mathfrak{c}}(x_2,x_3)) \arrow[dl, "m_{\mathfrak{c}}(\mathrm{id},t)"] \arrow[dr, "a_{\mathfrak{c}}^{-1}"] & \\
m_{\mathfrak{c}}(x_1,m_{\mathfrak{c}}(x_3,x_2)) \arrow[d, "a_{\mathfrak{c}}^{-1}"'] & & m_{\mathfrak{c}}(m_{\mathfrak{c}}(x_1,x_2),x_3) \arrow[d, "t"] \\
m_{\mathfrak{c}}(m_{\mathfrak{c}}(x_1,x_3),x_2) \arrow[dr, "m_{\mathfrak{c}}(t,\mathrm{id})"'] & & m_{\mathfrak{c}}(x_3,m_{\mathfrak{c}}(x_1,x_2)) \arrow[dl, "a_{\mathfrak{c}}^{-1}"] \\
& m_{\mathfrak{c}}(m_{\mathfrak{c}}(x_3,x_1),x_2) &
\end{tikzcd}$$

Figure 1.3.2: Hexagons

Recall that an algebra over a colored operad Q is a morphism $\mathsf{Q} \to \mathsf{End}_{(A,B)}$, where

$$\mathsf{End}^{\mathfrak{c}}_{(A,B)}(n,m) = \hom(B^{\otimes n} \otimes A^{\otimes m}, A), \quad \mathsf{End}^{\mathfrak{o}}_{(A,B)}(n,m) = \hom(B^{\otimes n} \otimes A^{\otimes m}, B).$$

Given a morphism $\mathsf{PaPB} \to \mathsf{End}_{(\mathsf{M},\mathsf{N})}$ with the names as in Theorem 1.3.3 for the images of the generators, the previous coherence diagrams are exactly the

$$
\begin{tikzcd}[column sep=tiny]
& m_o(m_o(f(x_1),f(x_2)),f(x_3)) \arrow[dl, "m_o(\pi,\mathrm{id})"'] \arrow[dr, "a_c"] & \\
m_o(f(m_c(x_1,x_2)),f(x_3)) \arrow[d, "\pi"'] & & m_o(f(x_1),m_o(f(x_2),f(x_3))) \arrow[d, "m_o(\mathrm{id},\pi)"] \\
f(m_c(m_c(x_1,x_2),x_3)) \arrow[dr, "f(a_c)"'] & & m_o(f(x_1),f(m_c(x_2,x_3))) \arrow[dl, "\pi"] \\
& f(m_c(x_1,m_c(x_2,x_3))) &
\end{tikzcd}
$$

Figure 1.3.3: F is monoidal

$$
\begin{tikzcd}[column sep=tiny]
& m_o(m_o(f(x_1),x_2),x_3) \arrow[dl, "m_o(\Psi,\mathrm{id})"'] \arrow[dr, "a_o"] & \\
m_o(m_o(x_2,f(x_1)),x_3) \arrow[d, "a_o"'] & & m_o(f(x_1),m_o(x_2,x_3)) \arrow[d, "\Psi"] \\
m_o(x_2,m_o(f(x_1),x_3)) \arrow[dr, "m_o(\mathrm{id},\Psi)"'] & & m_o(m_o(x_2,x_3),f(x_1)) \arrow[dl, "a_o"] \\
& m_o(x_2,m_o(x_3,f(x_1))) &
\end{tikzcd}
$$

Figure 1.3.4: Ψ is a half-braiding

$$
\begin{tikzcd}
m_o(f(x_1),f(x_2)) \arrow[r, "\Psi"] \arrow[d, "\pi"'] & m_o(f(x_2),f(x_1)) \arrow[d, "\pi"] \\
f(m_c(x_1,x_2)) \arrow[r, "f(t)"'] & f(m_c(x_2,x_1))
\end{tikzcd}
$$

Figure 1.3.5: F is braided

$$
\begin{tikzcd}[column sep=tiny]
& m_o(f(x_1),m_o(f(x_2),x_3)) \arrow[dl, "m(\mathrm{id},\Psi)"'] \arrow[dr, "a_o^{-1}"] & \\
m_o(f(x_1),m_o(x_3,f(x_2))) \arrow[ddd, "a_o^{-1}"'] & & m_o(m_o(f(x_1),f(x_2)),x_3) \arrow[d, "m_o(\pi,\mathrm{id})"] \\
& & m_o(f(m_c(x_1,x_2)),x_3) \arrow[d, "\Psi"] \\
& & m_o(x_3,f(m_c(x_1,x_2))) \arrow[d, "m_o(\mathrm{id},\pi^{-1})"] \\
m_o(m_o(f(x_1),x_3),f(x_2)) \arrow[dr, "m_o(\Psi,\mathrm{id})"'] & & m_o(x_3,m_o(f(x_1),f(x_2))) \arrow[dl, "a_o^{-1}"] \\
& m_o(m_o(x_3,f(x_1)),f(x_2)) &
\end{tikzcd}
$$

Figure 1.3.6: F is monoidal w.r.t. half-braidings

diagrams encoding the fact that $(m_\mathfrak{c}, a_\mathfrak{c}, t_\mathfrak{c})$ is a braided monoidal structure on M, $(m_\mathfrak{o}, a_\mathfrak{o})$ is a monoidal structure on N, and F is a braided monoidal functor to the Drinfeld center.

Corollary 1.3.4. *An algebra over* PaPB *is the data of a (non-unitary) monoidal category* $(\mathsf{N}, \otimes)$*, a (non-unitary) braided monoidal category* $(\mathsf{M}, \otimes, \tau)$*, and a strong braided monoidal functor* $F : \mathsf{M} \to \mathcal{Z}(\mathsf{N})$*.*

Definition 1.3.5. Between two objects $x, y \in \mathsf{PaPB}(n, m)$ such that the terrestrial (resp. aerial) points of x are numbered in the same order as the terrestrial (resp. aerial) points y, there is a unique morphism $\mu \in \hom_{\mathsf{PaPB}(n,m)}(x, y)$, called a **shuffle-type morphism**, such that the aerial strands do not cross each other (see Figure 1.3.7).

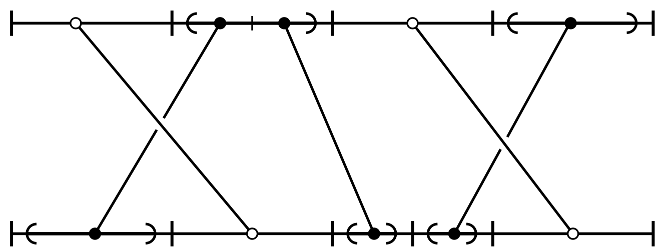


Figure 1.3.7: Example of shuffle-type morphism

Proof of Theorem 1.3.3. It is clear (a simple exercise in drawing braid diagrams) that the morphisms of PaPB satisfy the corresponding relations, thus we get the "only if" part of the theorem.

Let $Y \in \hom_{\mathsf{PaPB}(n,m)}(x_1, x_2)$ be a morphism. We want to decompose it as in Figure 1.3.8:

- We first arbitrarily choose two objects x_1', x_2' which are in the image of the product $\mathsf{PaP}(n) \times \mathsf{PaB}(m)$ by $\mu_\mathfrak{o}(-, f(-))$. In other words, $x_i' = \mu_\mathfrak{o}(x_i^\mathfrak{o}, f(x_i^\mathfrak{c}))$ is the concatenation of an object $x_i^\mathfrak{o} \in \mathsf{PaP}(n) = \mathsf{PaPB}(n, 0)$ and of the image by f of an object $x_i^\mathfrak{c} \in \mathsf{PaB}(m)$. We also require that the aerial points (resp. the terrestrial points) of x_i' are numbered in the same order as those of x_i.
- We take the unique shuffle-type morphism $\mu : x_1 \to x_1'$.
- We build a morphism $X = \mu_\mathfrak{o}(X^\mathfrak{o}, f(X^\mathfrak{c})) : x_1' \to x_2'$. It is the concatenation of $X^\mathfrak{o} \in \mathsf{PaP}(n)$, and $X^\mathfrak{c} \in \mathsf{PaB}(m)$. Explicitly, $X^\mathfrak{o}$ is the colored permutation where all the aerial strands of $\omega_*(Y)$ have been forgotten, and $X^\mathfrak{c}$ is the colored braid where all the terrestrial strands of $\omega_*(Y)$ have been forgotten.

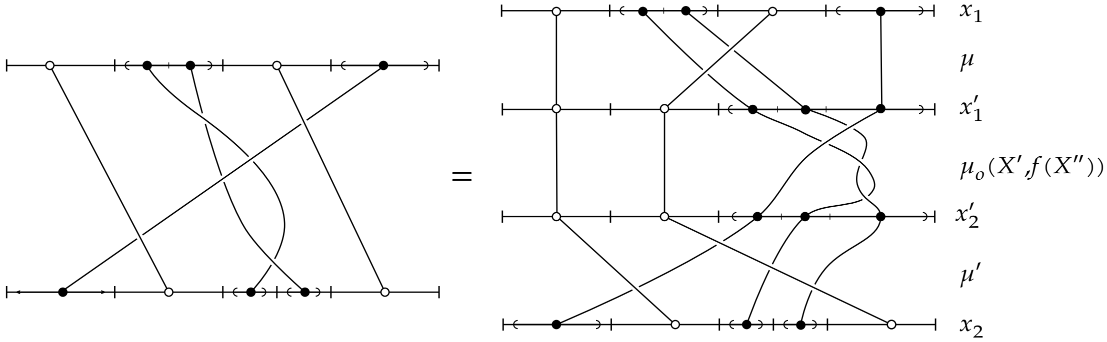


Figure 1.3.8: Decomposition in $\mathsf{PaPB}(2,3)$

- Finally, we take the unique shuffle-type morphism $\mu' : x'_2 \to x_2$.

By construction, $Y = \mu' \circ X \circ \mu$. Besides, this decomposition is unique given the specified intermediary objects x'_1, x'_2, so it suffices to show that θ can be defined unequivocally on each part, that it doesn't depend on the choice of x'_1 and x'_2, and that it is compatible with operadic composition.

The shuffle-type morphisms are all in the suboperad of PaPB generated by $\alpha_{\mathfrak{o}}^{\pm 1}$, $p^{\pm 1}$, $\psi^{\pm 1}$: first one can cut the objects of PaB in the smallest possible pieces with p^{-1}, the $\alpha_{\mathfrak{o}}^{\pm 1}$ and $\psi^{\pm 1}$ can be used to bring all the aerial points at their positions, and finally p is used to glue back all the aerial pieces. By the theorems I.6.1.7 and I.6.2.4 of [Fre17], the two morphisms $X^{\mathfrak{o}} \in \mathsf{PaP}(n)$ and $X^{\mathfrak{c}} \in \mathsf{PaB}(m)$ are respectively in the suboperads generated by $\mu_{\mathfrak{o}}^{\pm 1}, \alpha_{\mathfrak{o}}^{\pm 1}$ and by $\mu_{\mathfrak{c}}^{\pm 1}, \alpha_{\mathfrak{c}}^{\pm 1}, \tau^{\pm 1}$. It follows that every morphism Y of PaPB is in the suboperad generated by all these elements, thus the morphism $\theta : \mathsf{PaPB} \to \mathsf{P}$, if it exists, is unique.

By the same theorems of [Fre17], the pentagons (Figure 1.3.1) and the hexagons (Figure 1.3.2) show that the morphism θ can be defined with no ambiguity on the two pieces $X^{\mathfrak{c}}$ and $X^{\mathfrak{o}}$. The possible choices for x'_1 and x'_2 are all related by associators, so the pentagons (Figure 1.3.1) and MacLane's coherence theorem [Mac98] for monoidal categories show that the image does not depend on the choice of x'_1 and x'_2.

Let $\mu \in \mathsf{PaPB}(n,m)$ be a shuffle-type morphism; we saw that it could be decomposed in terms of $p^{\pm 1}$, $\psi^{\pm 1}$ and $\alpha^{\pm 1}$. The coherence theorem of MacLane [Mac98] and the coherence theorem of Epstein [Eps66] on monoidal functors (non-symmetric version) show that, thanks to the pentagons (Figure 1.3.1) and the fact that F is monoidal (Figure 1.3.3), the image $\theta(\mu)$ neither depends on the choice of associator decomposition, nor on the choice of decomposition of $p^{\pm 1}$, nor on the way the $\psi^{\pm 1}$ are gathered in the parenthesizing. It thus suffices to define θ on the underlying morphism of CoPB.

This last morphism is actually an element of the braid group B_{n+m} (of course not all the elements of the braid group can given a morphism: terrestrial strands cannot cross any other strand). By seeing ψ as a braiding, and by interpreting the relations of Figures 1.3.4 and 1.3.6 as two hexagon relations, we can adapt the proof of the step 2 of [Fre17, Theorem 6.2.4] to see that the image by θ of this braid does not depend on its representation. Finally, θ is well-defined on every morphism. It remains to show that it respects operad composition.

By adapting the fourth step of the proof of the same theorem of [Fre17] and by using the relation of Figure 1.3.5, we can see that θ respects operadic composition. Indeed, shuffle-type morphism are sent by construction on elements decomposed in terms of associators, their inverses, $p^{\pm 1}$ and $\psi^{\pm 1}$, while for example $\psi \circ_1^{\mathfrak{o}} \mathrm{id}_f = f(\tau)$; but thanks to the relation of Figure 1.3.5, both elements are equal in the image. □

By dropping all mentions of parenthesizing, we get:

Proposition 1.3.6. *An algebra over* CoPB *consists of a strict non-unitary monoidal category* N*, a strict braided non-unitary monoidal category* M*, and of a strict braided monoidal functor* $F : \mathsf{M} \to \mathcal{Z}(\mathsf{N})$.

1.3.2 Unitary versions

We are going to define unitary versions CoPB_+ and PaPB_+ of the operads we are studying, satisfying $\mathsf{CoPB}_+(0,0) = \mathsf{PaPB}_+(0,0) = \{*_{\mathfrak{o}}\}$. For consistency we will denote $*_{\mathfrak{c}}$ the element of the one-colored unitary operads we will consider (CoB_+, PaB_+, etc.).

Definition 1.3.7. Let CoPB_+ be the relative operad over CoB_+, defined as a unitary extension of CoPB_+. Composition with $*_{\mathfrak{c}} \in \mathsf{CoB}_+(0)$ forgets aerial strands, while composition with $*_{\mathfrak{o}}$ forgets terrestrial strands.

Definition 1.3.8. Let $\Omega\Omega_+$ be the relative operad over Ω_+, a unitary extension of $\Omega\Omega$. Composition with nullary elements is given on generators by (it is not necessary to specify $f(*_{\mathfrak{o}})$ as $\Omega\Omega_+(0,0)$ is a singleton anyway):

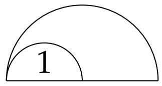
$$\mu_{\mathfrak{c}}(*_{\mathfrak{c}}, \mathrm{id}_{\mathfrak{c}}) = \mu_{\mathfrak{c}}(\mathrm{id}_{\mathfrak{c}}, *_{\mathfrak{c}}) = \mathrm{id}_{\mathfrak{c}}, \quad \mu_{\mathfrak{o}}(*_{o}, \mathrm{id}_{\mathfrak{o}}) = \mu_{\mathfrak{o}}(\mathrm{id}_{\mathfrak{o}}, *_{\mathfrak{o}}) = \mathrm{id}_{\mathfrak{o}}\,.$$

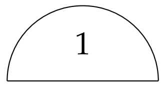
Let also $\mathsf{PaPB}_+ = \omega_+^* \mathsf{CoPB}_+$ be the pullback of CoPB_+ along ω_+, where ω_+ is defined as the ω of Definition 1.2.8 (it is compatible with the unitary extensions).

Proposition 1.3.9. *There is a zigzag of categorical equivalences, where* $\Omega\Omega'_+ \subset \mathsf{SC}_+$ *is the sub–operad generated by* $m_{\mathfrak{c}}$*,* $m_{\mathfrak{o}}$*,* f *and the nullary elements:*

$$\pi\mathsf{SC}_+ \xleftarrow{\sim} \pi\mathsf{SC}_+|_{\Omega\Omega'_+} \xrightarrow{\sim} \mathsf{PaPB}_+ := \omega_+^*\mathsf{CoPB}_+ \xrightarrow{\sim} \mathsf{CoPB}_+,$$

Remark 1.3.10. In $\Omega\Omega'_+$, we have for example:

$\mu_o(\mathrm{id}_o, *_o) =$ 

$\neq$ 

$= \mathrm{id}_o,$

but $\mu_o(*_o, *_o) = *_o$. In other words, $*_o$ (and similarly $*_c$ for μ_c) is not a strict unit for $*_o$, but is still idempotent.

Proof. There is an evident morphism $\omega'_+ : \Omega\Omega'_+ \to \Omega\Omega_+$ sending generators on generators, and we check directly that $\pi\mathsf{SC}_+|_{\Omega\Omega'_+}$ is identified with the pullback $\omega'_+\mathsf{PaPB}'$. Since ω_+ and ω'_+ are both surjective, we obtain the two categorical equivalences

$$\pi\mathsf{SC}_+|_{\Omega\Omega'_+} \xrightarrow{\sim} \mathsf{PaPB}_+ \xrightarrow{\sim} \mathsf{CoPB}_+.$$

And since $\Omega\Omega'_+$ meets all connected components of SC_+ we also have that the inclusion $\pi\mathsf{SC}_+|_{\Omega\Omega'_+} \hookrightarrow \pi\mathsf{SC}_+$ is a categorical equivalence. □

The proof of the following proposition is a direct unitary extension of the proof of Corollary 1.3.4 (one also needs to extend the definition of the Drinfeld center to unitary monoidal categories, cf. the given references):

Proposition 1.3.11. *An algebra over* PaPB_+ *is given by:*

- *a monoidal category* $(\mathsf{N}, \otimes, 1_\mathsf{N})$ *with a strict unit;*
- *a braided monoidal category* $(\mathsf{M}, \otimes, 1_\mathsf{M}, \tau)$ *with a strict unit;*
- *a monoidal functor* $F : \mathsf{M} \to \mathcal{Z}(\mathsf{N})$ *satisfying* $F(1_\mathsf{M}) = 1_\mathsf{N}$.

An algebra over CoPB_+ *is given by the same data, but where the two tensors products are strictly associative, strictly braided for the second, and the functor is strict braided monoidal.*

1.4 Chord diagrams

Let P an operad in groupoids. Its completion $\hat{\mathsf{P}}$ is defined by the Malcev completion arity by arity:

$$\hat{\mathsf{P}}(r) = \mathbb{G}\widehat{\mathbb{Q}[\mathsf{P}(r)]},$$

and it is an operad in complete groupoids [Fre17, §I.9]. Here, $\mathbb{Q}[G]$ is the Hopf algebroid of the groupoid G; it has the same objects as G, and $\hom_{\mathbb{Q}[G]}(x, y) = \mathbb{Q}[\hom_G(x, y)]$ is the free $\mathbb{Q}$-module on the hom-set, equipped with a coalgebra structure where every generator is grouplike. It is completed at the augmentation ideal, and then the functor $\mathbb{G}$ extracts the grouplike elements to define an operad in complete groupoids. This completion is equipped with a canonical completion morphism $\mathsf{P} \to \hat{\mathsf{P}}$.

Definition 1.4.1. A morphism of operads in groupoids $\mathsf{P} \to \mathsf{Q}$ is called a **rational categorical equivalence** (denoted $\mathsf{P} \xrightarrow{\sim_{\mathbb{Q}}} \mathsf{Q}$) if the induced morphism $\hat{\mathsf{P}} \to \hat{\mathsf{Q}}$ is a categorical equivalence. We write $\mathsf{P} \sim_{\mathbb{Q}} \mathsf{Q}$ if P and Q can be connected by a zigzag of rational categorical equivalences.

This definition is motivated by the following remark: if A is an abelian group, then $\widehat{A} = A \otimes_{\mathbb{Z}} \mathbb{Q}$. Examples of rational categorical equivalences include categorical equivalences and the canonical completion morphisms $\mathsf{P} \to \hat{\mathsf{P}}$. We refer to [Fre17, §I.9] for more details.

1.4.1 Drinfeld associators and chord diagrams

Definition 1.4.2. The Drinfeld–Kohno operad $\mathfrak{p}$ is an operad in Lie algebras.[2] In each arity we have the presentation by generators and relations:

$$\mathfrak{p}(r) = \mathtt{Lie}(t_{ij})_{1 \le i \ne j \le r} \; / \; ([t_{ij}, t_{kl}], [t_{ik}, t_{ij} + t_{jk}]),$$

and operadic composition is given by explicit formulas [Fre17, §I.10.2].

The universal enveloping algebra functor $\mathbb{U}$ being monoidal, $\mathbb{U}\mathfrak{p}$ is an operad in associative algebras. The algebra $\mathbb{U}\mathfrak{p}(r)$ is generated by chord diagrams with r strands, and composition is given by insertion of a diagram (cf. ibid. for precise definitions). We can complete $\mathfrak{p}$ with respect to the weight grading (the weight of t_{ij} is defined to be 1) to get an operad $\hat{\mathfrak{p}}$ in complete Lie algebras, and we can consider its completed universal enveloping algebra:

Definition 1.4.3. The **operad of completed chord diagrams**, $\widehat{\mathsf{CD}}$, is the operad in groupoids given by $\operatorname{ob} \widehat{\mathsf{CD}}(r) = *$ and $\operatorname{Hom}_{\widehat{\mathsf{CD}}(r)}(*, *) = \mathbb{G}\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)$. Operadic composition is induced by the one of $\mathfrak{p}$.

These operads have unitary extensions: restriction operations forget strands of the chord diagrams, and if a chord was attached to the strand, the diagram is sent to 0. We thus get unitary operads $\mathfrak{p}_+$, $\hat{\mathfrak{p}}_+$, and $\widehat{\mathsf{CD}}_+$.

Definition 1.4.4. A **Drinfeld associator** (with parameter $\mu \in \mathbb{Q}^\times$) is a morphism $\phi : \mathsf{PaB}_+ \to \widehat{\mathsf{CD}}_+$ of operads that sends the braiding $\tau \in \mathsf{PaB}_+(2)$ to $e^{\mu t_{12}/2} \in \widehat{\mathsf{CD}}_+(2)$. We let $\mathrm{Ass}^\mu(\mathbb{Q})$ be the set of such associators.

[2] The monoidal product in the category of Lie algebras is the direct sum.

If $\phi \in \mathrm{Ass}^{\mu}(\mathbb{Q})$, then the formal series in two variables

$$\Phi(t_{12}, t_{13}) := \phi(\alpha_{\mathfrak{c}}) \in \widehat{\mathsf{CD}}_+(3) \cong \mathbb{Q}[[t_{12}, t_{13}]]$$

is a Drinfeld associator in the usual sense, satisfying the usual equations (pentagon, hexagon), and vice versa. A Drinfeld associator ϕ extends to a categorical equivalence $\phi : \widehat{\mathsf{PaB}}_+ \xrightarrow{\sim} \widehat{\mathsf{CD}}_+$, i.e. ϕ is a rational equivalence. The set $\mathrm{Ass}^{\mu}(\mathbb{Q})$ is a torsor under the action of the Grothendieck–Teichmüller group $GT^1(\mathbb{Q})$, the group of automorphisms of $\widehat{\mathsf{PaB}}_+$ fixing $\mu_{\mathfrak{c}}$ and τ. A theorem of Drinfeld [Dri90] states that the set of associators $\mathrm{Ass}^1(\mathbb{Q})$ is nonempty.

We can also consider the operad $\widehat{\mathsf{PaCD}}_+$, which is the pullback of $\widehat{\mathsf{CD}}_+$ along the terminal morphism $\Omega \to \mathrm{ob}\,\widehat{\mathsf{CD}}_+ = *$. It is used to define the pro-unipotent version of the Grothendieck–Teichmüller group $GRT^1(\mathbb{Q})$, under which $\mathrm{Ass}^{\mu}(\mathbb{Q})$ is a pro-torsor. We recall the following statement [Fre17], which is actually a general fact about pullback along morphisms from a free operad: each morphism $\phi : \widehat{\mathsf{PaB}}_+ \to \widehat{\mathsf{CD}}_+$ admits a unique lifting

$$\begin{tikzcd}
 & \widehat{\mathsf{PaCD}}_+ \arrow[d] \\
\widehat{\mathsf{PaB}}_+ \arrow[ur, dashed, "\tilde{\phi}_+"] \arrow[r, "\phi"] & \widehat{\mathsf{CD}}_+
\end{tikzcd}$$

which is given by the identity on the level of objects. If the morphism ϕ came from a Drinfeld associator, then this defines an isomorphism of operads in groupoids:

$$\tilde{\phi}_+ : \widehat{\mathsf{PaB}}_+ \xrightarrow{\cong} \widehat{\mathsf{PaCD}}_+.$$

1.4.2 Shuffle of operads

By analogy with the decomposition of Figure 1.3.8, we define a new rational model in groupoids for $\pi\mathsf{SC}_+$ that involves the operad of chord diagrams.

Other description of CoPB_+

Definition 1.4.5. Let Π be the permutation operad: $\Pi(n) = \Sigma_n$, and operadic composition is given by bloc composition of permutations. We also denote Π the same operad seen as an operad in discrete groupoids. We also let Π_+ be its obvious unitary extension.

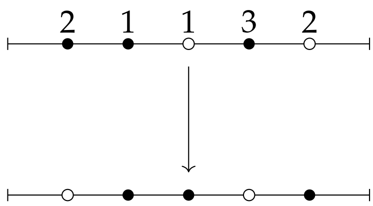
The following operad is meant to represent the shuffle-type morphisms of Definition 1.3.5:

Definition 1.4.6. We define the relative (unitary) operad in groupoid Sh_+ over Π_+. The set of objects of $\mathsf{Sh}_+(n,m)$ is $\mathrm{Sh}_{n,m} \times \Sigma_n \times \Sigma_m$, the same as CoPB_+ (with the same graphical representation). Operadic composition on the object level is the same as that of CoPB_+. On the level of morphisms:

$$\mathrm{Hom}_{\mathsf{Sh}_+(n,m)}((\mu,\sigma,\sigma'),(\nu,\tau,\tau')) = \begin{cases} * & \sigma = \tau, \sigma' = \tau', \\ \emptyset & \text{otherwise,} \end{cases}$$

and we check that this gives a well-defined relative operad over Π_+ (i.e. there are no maps $* \to \emptyset$ to define, and all the maps $\cdot \to *$ are terminal maps).

Graphically, we simply represent morphisms of Sh_+ by an arrow between two bicolored configurations on the interval. Such an arrow exists iff the terrestrial (resp. aerial) points of the first configuration are in the same order as the terrestrial (resp. aerial) points of the second configuration, so we do not write the labels for the second configuration:



Remark 1.4.7. The symmetric groups Σ_n and Σ_m act on the left *and* on the right on $\mathsf{Sh}_+(n,m)$ (by multiplication on respective factors).

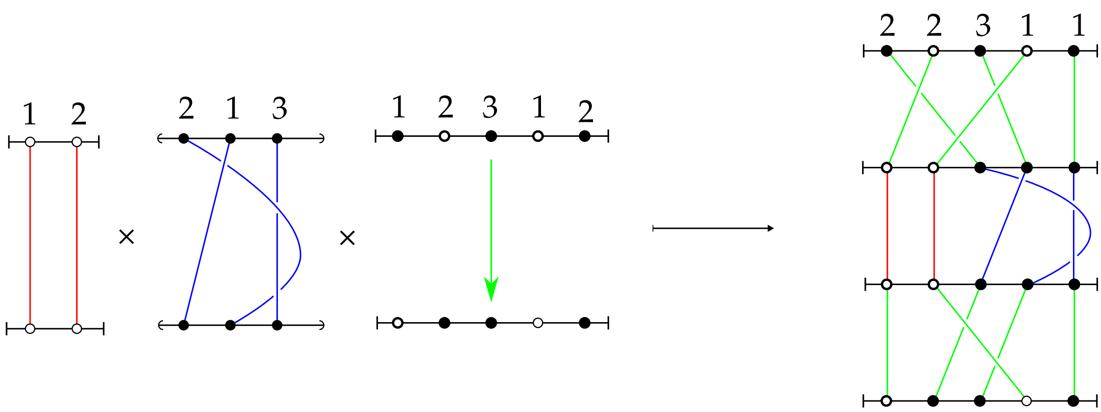
Lemma 1.4.8. *The groupoid* $\mathsf{CoPB}_+(n,m)$ *is isomorphic to*

$$(\mathsf{CoP}_+(n) \times \mathsf{CoB}_+(m)) \times_{\Sigma_n \times \Sigma_m} \mathsf{Sh}_+(n,m).$$

Proof. We define:

$$\zeta : (\mathsf{CoP}_+(n) \times \mathsf{CoB}_+(m)) \times_{\Sigma_n \times \Sigma_m} \mathsf{Sh}_+(n,m) \to \mathsf{CoPB}_+(n,m)$$

by a graphical calculus:



Concretely, on objects, we define:

$$\begin{aligned}
\zeta : \mathrm{ob}(\mathsf{CoP}_+(n) \times \mathsf{CoB}_+(m) \times_{\Sigma_n \times \Sigma_m} \mathsf{Sh}_+(n,m)) \\
\xrightarrow{=} (\Sigma_n \times \Sigma_m) \times_{\Sigma_n \times \Sigma_m} \mathrm{ob}\,\mathsf{CoPB}(n,m) \\
\xrightarrow{\cong} \mathrm{ob}\,\mathsf{CoPB}(n,m).
\end{aligned}$$

On morphisms, $\zeta[u,x,\mu]$ (for $u \in \mathsf{CoP}_+(n)$, $x \in \mathsf{CoB}_+(m)$, $\mu \in \mathsf{Sh}_+(n,m)$) is the composition of the unique shuffle type morphism that brings all terrestrial points to the left, then the concatenation of u and x, then the unique shuffle-type morphism that brings ground point to their places. We thus get a well-defined (up to isotopy) braid, and it is easy to see that this gives a bijection on morphisms. $\square$

On can thus transport the operadic composition, which will serve as inspiration for Equation (1.4.15) to come.

A variation on PaPB_+

We first define a new operad PaPB'_+, a minor variation on PaPB_+.

Definition 1.4.9. Let $\omega_+ : \Omega\Omega_+ \to \mathrm{ob}\,\mathsf{CoPB}_+ \cong \mathrm{ob}\,\mathsf{Sh}_+$ be the morphism of Definition 1.2.8. We define PaSh_+ to be the pullback of Sh_+ along ω.

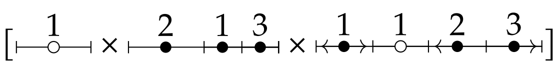
Remark 1.4.10. There is a function of sets $U : \mathrm{ob}\,\mathsf{PaPB}_+(0,m) \to \mathrm{ob}\,\mathsf{PaB}_+(m)$ that forgets the second level of parenthesizing.

Definition 1.4.11. Let

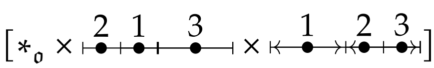
$$\mathsf{PaPB}'_+(n,m) \subset (\mathsf{PaP}_+(n) \times \mathsf{PaB}_+(m)) \times_{\Sigma_n \times \Sigma_m} \mathsf{PaSh}_+(n,m),$$

be the full subgroupoid whose objects $[u,x,\mu]$ such that there exists a permutation $\tau \in \Sigma_m$ satisfying $U(\mu(*_0,\dots,*_0)) = x \cdot \tau$ (this does not depend on the choice of a representative for the coinvariants).

Example 1.4.12. For example,



$\in \mathrm{ob}\,\mathsf{PaPB}'(1,3)$, but



$\notin \mathrm{ob}\,\mathsf{PaPB}'(0,3)$.

Lemma 1.4.13. *The symmetric sequence $\mathsf{PaPB}'_+(n)$ is a right module over PaB_+, given by:*

$$\begin{aligned}
\circ_i^{\mathfrak{c}} : \mathsf{PaPB}'_+(n,m) \times \mathsf{PaB}_+(k) &\to \mathsf{PaPB}'_+(n, m+k-1) \\
[u,x,\mu] \times y &\mapsto [u, x \circ_i y, \mu \circ_i^{\mathfrak{c}} 1_{\Sigma_k}],
\end{aligned}$$

where 1_{Σ_k} is seen as a morphism in $\omega^* \Pi_+(k)$ between the source of y and the target of y.

Proof. The operad PaB_+ is a right module over itself, and $\mathsf{PaSh}_+(n)$ is a right module over Ω_+. One can then directly check that the above formula defines a right module structure over PaB_+. $\square$

Definition 1.4.14. Let P be an operad in some symmetric monoidal category. The **shifted operad** $\mathsf{P}[\cdot]$ [Fre09, §10.1] is an operad in right modules over P (i.e. an operad relative over P), given by $\mathsf{P}[n](m) = \mathsf{P}(n+m)$, and the structure maps are induced by the operad structure of P (cf. ibid. for explicit formulas).

To define the operad structure of PaPB'_+, we first define a morphism of colored collections $\rho : \mathsf{PaPB}'_+ \to \mathsf{PaB}[\cdot]$, that will be similar to the definition of ζ 1.4.8. It is again defined in a graphical way, see Figure 1.4.1 (the starred numbers correspond to shifted entries). The precise definition involves the inclusion $\iota : \mathsf{PaP}_+ \hookrightarrow \mathsf{PaB}_+$, concatenation, the functor $\mathsf{PaSh}_+ \to \mathsf{PaB}_+$ (picking the unique shuffle-type morphism when it exists), $\eta : \mathsf{PaP}_+ \to \mathsf{PaP}_+[\cdot]$ (where $\eta : \mathsf{PaP}_+(n) \cong \mathsf{PaP}_+[n](0)$), as well as U on objects – see Equation (1.4.20).

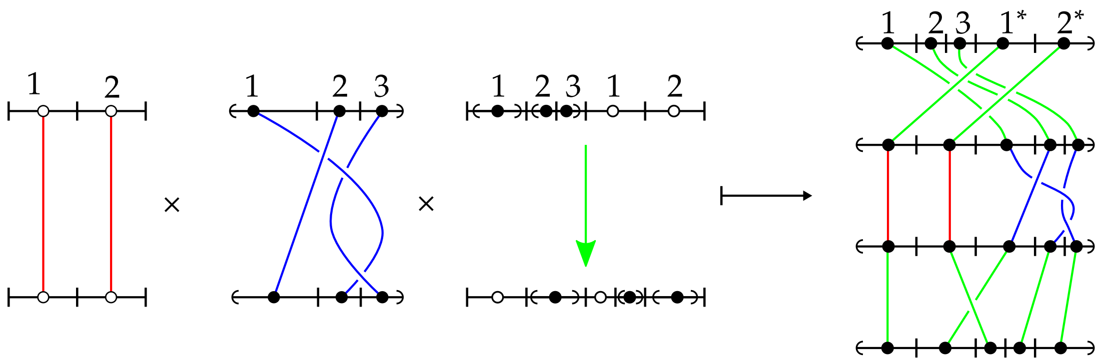


Figure 1.4.1: Definition of $\rho : \mathsf{PaPB}'_+(2,3) \to \mathsf{PaB}_+[2](3)$

The operad structure is then defined by (where $\sigma \in \Sigma_m$ is such that

$$U(\mu(*_{\mathfrak{o}}, \dots, *_{\mathfrak{o}})) = x \cdot \sigma,$$

and where y_i is seen as an element of $\mathsf{PaPB}'_+[0](l_i)$):

$$\begin{aligned}\gamma : \mathsf{PaPB}'_+(r,s) \times \mathsf{PaPB}'_+(k_1,l_1) \times \cdots \times \mathsf{PaPB}'_+(k_r,l_r) &\to \mathsf{PaPB}'_+(\textstyle\sum k_i, s + \sum l_i)\\ [u,x,\mu] \times [v_1,y_1,\sigma_1] \times \cdots \times [v_r,y_r,\sigma_r] &\\ \mapsto [u(v_1,\dots,v_r), \sigma^{-1} \cdot \underbrace{\rho[u,x,\mu]}_{\mathsf{PaPB}'_+[r](s)}(y_1,\dots,y_r), \mu(\sigma_1,\dots,\sigma_r)] &\end{aligned} \tag{1.4.15}$$

and where the identity of the operad is $\mathrm{id} = [\overset{1}{\vdash\!\circ\!\dashv} \times *_{\mathfrak{c}} \times \overset{1}{\vdash\!\circ\!\dashv}] \in \mathsf{PaPB}'_+(1,0)$.

Proposition 1.4.16. *Given this operadic composition, this identity and this right module structure,* PaPB'_+ *is an operad relative over* PaB_+.

Proof. The σ^{-1} in the formula ensures that this γ is well-defined (it does not depend on the representative in the coinvariants). The fact that γ is equivariant is a direct consequence of the fact that the operad structures of PaP_+, PaSh_+ and $\mathsf{PaB}_+[\cdot]$ are equivariant.

Let $[u,x,\mu] \in \mathsf{PaPB}'_+(n,m)$. The identity $\mathrm{id}([u,x,\mu]) = [u,x,\mu]$ is immediate by definition, and from the condition on the objects of PaPB'_+, one can also show the identity $[u,x,\mu](\mathrm{id},\dots,\mathrm{id}) = [u,x,\mu]$.

To see that γ is a morphism of right PaB_+-modules, it is enough to have the identity

$$\rho[u,x,\mu](y_1,\dots,y_j \circ_i z,\dots,y_r) = \rho[u,x,\mu](y_1,\dots,y_r) \circ_{l_1+\dots+l_{j-1}+i} z$$

for $z \in \mathsf{PaB}_+(m)$; but since $\mathsf{PaB}_+[\cdot]$ is an operad, this identity is satisfied.

Finally, associativity of γ follows from the condition on objects (to show that $\mathrm{ob}\,\gamma$ is associative), and from the fact that $\mathsf{PaB}_+[\cdot]$ is an operad (to show associativity on morphisms). □

Proposition 1.4.17. *There exists a categorical equivalence* $\mathsf{PaPB}'_+ \xrightarrow{\sim} \mathsf{CoPB}_+$.

Proof. This equivalence is given in arity (n,m) by the restriction to $\mathsf{PaPB}'_+(n,m)$ of the composite:

$$\begin{aligned}
&(\mathsf{PaP}_+(n) \times \mathsf{PaB}_+(m)) \times_{\Sigma_n \times \Sigma_m} \mathsf{PaSh}_+(n,m) \\
&\to (\mathsf{CoP}_+(n) \times \mathsf{CoB}_+(m)) \times_{\Sigma_n \times \Sigma_m} \mathsf{Sh}_+(n,m) \\
&\xrightarrow[\zeta]{\cong} \mathsf{CoPB}_+(n,m)
\end{aligned}$$

By construction (the operad structure of PaPB'_+ is directly mimicked from the operad structure of CoPB_+), this yields a morphism of operads $\mathsf{PaPB}'_+ \to \mathsf{CoPB}_+$. Since $\mathsf{PaP}_+ \to \mathsf{CoP}_+$, $\mathsf{PaB}_+ \to \mathsf{CoB}_+$ and $\mathsf{PaSh}_+ \to \mathsf{Sh}_+$ are categorical, their product is too, thus the above morphism yields a categorical equivalence. □

An operad defined from chord diagrams

We choose a Drinfeld associator $\phi : \mathsf{PaB}_+ \to \widehat{\mathsf{CD}}_+$; let $\tilde{\phi}_+ : \mathsf{PaB}_+ \to \widehat{\mathsf{PaCD}}_+$ be its unique lifting. Similarly to the definition of PaPB'_+, we will define a relative operad $\mathsf{Pa}\widehat{\mathsf{PCD}}^{\phi}_+$ over $\widehat{\mathsf{PaCD}}_+$, that will combine parenthesized shuffles, parenthesized permutations and parenthesized chords diagrams.

Definition 1.4.18. Let

$$\mathsf{PaP}\widehat{\mathsf{CD}}^{\phi}_{+}(n,m) \subset (\mathsf{PaP}_{+}(n) \times \widehat{\mathsf{PaCD}}_{+}(m)) \times_{\Sigma_n \times \Sigma_m} \mathsf{PaSh}_{+}(n,m)$$

be the full subgroupoid whose objects are classes $[u,\alpha,\mu]$ such that there exists $\sigma \in \Sigma_m$ satisfying $U(\mu(*_0,\dots,*_0)) = \alpha \cdot \sigma$.

Remark 1.4.19. The objects of $\mathsf{PaP}\widehat{\mathsf{CD}}^{\phi}_{+}$ are the same as the objects of PaPB'_{+}.

We also define a morphism $\rho_\phi : \mathsf{PaP}\widehat{\mathsf{CD}}^{\phi}_{+} \to \widehat{\mathsf{PaCD}}_{+}[\cdot]$. Its definition is similar to that of Figure 1.4.1, but one cannot directly use graphical calculus anymore. In the picture, concatenation in PaB_{+} and $\mathsf{PaB}_{+}[\cdot]$ corresponded to $m_{\mathfrak{c}}$. Pre- and post-composition by shuffles in PaPB'_{+} came from a morphism of operads σ : $\mathsf{PaSh}_{+} \to \mathsf{PaB}_{+}[\cdot]$. We have $\operatorname{ob}\sigma = \mathrm{id}$, and $\sigma(\mu \to \mu')$, denoted $\sigma^{\mu}_{\mu'}$ to simplify, is the unique morphism $\mathsf{PaB}[\cdot]$ of shuffle-type between the corresponding objects.

We also recall the canonical morphism $\eta : \mathsf{P} \to \mathsf{P}[\cdot]$, given in arity m by $\mathsf{P}(m) \cong \mathsf{P}[m](0)$. Finally, we could define ρ by the following formula (where $x \in \mathsf{PaB}_{+}(m)$ is identified with $x \in \mathsf{PaB}_{+}[0](m)$):

$$\rho[u,x,\mu] = \sigma^{m_{\mathfrak{c}}(\iota(\eta(\mathrm{tgt}(u))),\mathrm{tgt}(x))}_{\mathrm{tgt}(\mu)} \circ m_{\mathfrak{c}}(\iota(\eta(u)),x) \circ \sigma^{\mathrm{src}(\mu)}_{m_{\mathfrak{c}}(\iota(\eta(\mathrm{src}(u))),\mathrm{src}(x))}. \tag{1.4.20}$$

By analogy, we define:

$$\rho_\phi : \mathsf{PaP}\widehat{\mathsf{CD}}^{\phi}_{+}(n,m) \to \widehat{\mathsf{PaCD}}_{+}[n](m)$$

To simplify, we let $\tilde{\phi}_{+}(m_{\mathfrak{c}}) = \tilde{m}_{\mathfrak{c}}$, $\tilde{\phi}_{+} \circ \sigma = \tilde{\sigma}$, $\tilde{\phi}_{+}\iota\eta = \tilde{\iota}$, and we again identify $\alpha \in \widehat{\mathsf{PaCD}}_{+}(m)$ with $\alpha \in \widehat{\mathsf{PaCD}}_{+}[0](m)$. Then ρ_ϕ is given by:

$$[u,\alpha,\mu] \mapsto \tilde{\sigma}^{\tilde{m}_{\mathfrak{c}}(\tilde{\iota}(\mathrm{tgt}(u)),\mathrm{tgt}(\alpha))}_{\mathrm{tgt}(\mu)} \circ \tilde{m}_{\mathfrak{c}}(\tilde{\iota}(u),\alpha) \circ \tilde{\phi}_{+}(\tilde{\sigma}^{\mathrm{src}(\mu)}_{\tilde{m}_{\mathfrak{c}}(\tilde{\iota}(\mathrm{src}(u)),\mathrm{src}(\alpha))}).$$

Graphically, ρ_ϕ looks like Figure 1.4.2, where the gray boxes represent applications of the associator. We then define an operadic composition in a similar manner to Equation (1.4.15), replacing ρ by ρ_ϕ. We also define a right $\widehat{\mathsf{PaCD}}_{+}$-module similar to that of PaPB'_{+}.

Theorem 1.4.21. *The data $\mathsf{PaP}\widehat{\mathsf{CD}}^{\phi}_{+}$, equipped with these structures, is a relative operad over $\widehat{\mathsf{PaCD}}_{+}$, and the morphism $\mathsf{PaPB}'_{+} \to \mathsf{PaP}\widehat{\mathsf{CD}}^{\phi}_{+}$ induced by $\tilde{\phi}_{+}$ is a rational categorical equivalence of operads. There is thus a zigzag:*

$$\pi\mathsf{SC}_{+} \xleftarrow{\sim} (\pi\mathsf{SC}_{+})_{\Omega\Omega'_{+}} \xrightarrow{\sim} \mathsf{PaPB}_{+} \xrightarrow{\sim} \mathsf{CoPB}_{+} \xleftarrow{\sim} \mathsf{PaPB}'_{+} \xrightarrow{\sim_{\mathbb{Q}}} \mathsf{PaP}\widehat{\mathsf{CD}}^{\phi}_{+}.$$

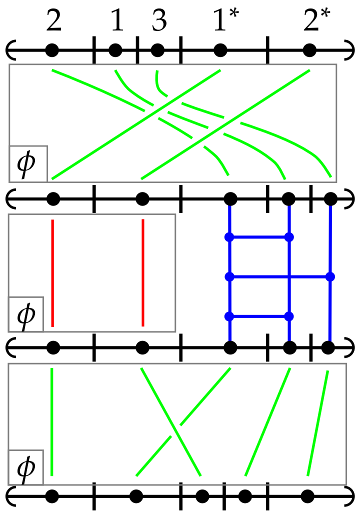


Figure 1.4.2: Graphical representation of ρ_ϕ

Proof. The proof that $\mathsf{PaP}\widehat{\mathsf{CD}}^{\phi}_{+}$ is a relative operad is identical to the proof of Proposition 1.4.16, and the fact that the morphism induced by $\tilde{\phi}_+$ is a morphism of operads follows by a direct inspection of the definitions.

The morphism $\tilde{\phi}_+ : \mathsf{PaB}_+ \to \widehat{\mathsf{PaCD}}_+$ is not a categorical equivalence, but it factors as:

$$\begin{tikzcd} \mathsf{PaB}_+ \arrow[rr, "\tilde{\phi}_+"] \arrow[dr, "\sim_{\mathbb{Q}}"'] & & \widehat{\mathsf{PaCD}}_+ \\ & \widehat{\mathsf{PaB}}_+ \arrow[ur, dashed, "\cong"'] & \end{tikzcd}$$

where the morphism $\widehat{\mathsf{PaB}}_+ \to \widehat{\mathsf{PaCD}}_+$ is an isomorphism of operads in groupoids. It follows that $\tilde{\phi}_+$ is a rational equivalence of operads in groupoids, thus $\mathsf{PaPB}'_+ \to \mathsf{PaP}\widehat{\mathsf{CD}}^{\phi}_{+}$ is also a rational categorical equivalence. By combining this fact with Propositions 1.3.9 and 1.4.17, we finally get the zigzag of the theorem. □

1.4.3 Non-formality

A theorem of Livernet [Liv15, Theorem 3.1] states that the Swiss-Cheese operad is not *formal*: its homology $H_*(\mathsf{SC})$ is not equivalent to its operad of chains $C_*(\mathsf{SC})$. We give an interpretation of this fact here.

We consider a stronger version of formality, which involves the models of rational homotopy theory of Sullivan (see [Fre17, §II] for the applications to operads). Let

$$\langle - \rangle^{\mathbb{L}} : \mathsf{CDGA}_+^{\mathrm{op}} \to s\mathsf{Set}$$

be the *derived* Sullivan realization functor,[3] that uses commutative dg-algebras as rational models for spaces.

We rely on *co*homological models to study rational homotopy theory, so we consider dual structures of our objects and we use cooperads rather than operads. For example, the cooperad $\mathtt{Com}^*$ governing cocommutative coalgebras is dual to the operad $\mathtt{Com}$ governing commutative algebras.

To encode the rational homotopy-theoretic information, we add commutative structures to our objects, and we consider **Hopf cooperads**, i.e. cooperads in the category of commutative algebras. The Sullivan realization of a Hopf cooperad is a simplicial operad.

Splitting of $H_*(\mathsf{SC})$ as a Voronov product

A theorem of Cohen [Coh76] describes the homology of the little disks operads $\mathsf{e}_n := H_*(\mathsf{D}_n)$. In low dimensions, $\mathsf{e}_1 \cong \mathtt{Ass}$ is the operad governing associative algebras, while $\mathsf{e}_2 \cong \mathtt{Ger}$ is the operad governing Gerstenhaber algebras. These are **Hopf operads** (i.e. operads in the category of cocommutative coalgebras): the coproduct of the product of either $\mathtt{Ass}$ or $\mathtt{Ger}$ is $\Delta(\mu) = \mu \otimes \mu$, while the coproduct of the bracket of $\mathtt{Ger}$ is $\Delta(\lambda) = \mu \otimes \lambda + \lambda \otimes \mu$. Their duals $\mathtt{Ass}^*$ and $\mathtt{Ger}^*$ are Hopf cooperads.

The homology of the Swiss-Cheese operad $\mathtt{sc} := H_*(\mathsf{SC})$ governs the action of a Gerstenhaber algebra on an associative algebra. A theorem of Voronov [Vor99, Theorem 3.3] (see also [HL12, Theorem 6.1.1] for this particular variant) states that an algebra over $H_*(\mathsf{SC})$ is a triple (B, A, f) where B is a Gerstenhaber algebra, A is an associative algebra, and $f : B \to A$ is a central morphism of associative algebras, which thus makes A into an associative algebra over the commutative algebra B. Let us note that Corollary 1.3.4 is a categorical analogue of Voronov's theorem, the Drinfeld center of a monoidal category replacing the center of an associative algebra.

This theorem can be interpreted in the following way.

Definition 1.4.22. Given two operads P and Q and a morphism $\mathtt{Com} \to \mathsf{P}$, one can define the **Voronov product** $\mathsf{P} \otimes \mathsf{Q}$ [Vor99]. It is a relative operad over P, defined by $(\mathsf{P} \otimes \mathsf{Q})(n, m) = \mathsf{P}(m) \otimes \mathsf{Q}(n)$. Insertion of a closed-output operation:

$$\circ_i^{\mathfrak{c}} : (\mathsf{P}(m) \otimes \mathsf{Q}(n)) \otimes \mathsf{P}(m') \to \mathsf{P}(m + m' - 1) \otimes \mathsf{Q}(n)$$

[3] The underived realization functor maps a commutative, unitary differential graded algebra A to the simplicial set $\langle A \rangle = \hom_{\mathsf{CDGA}_+}(A, \Omega^*_{PL}(\Delta^\bullet))$. The derived version takes a cofibrant replacement first.

uses the operad structure of P, while insertion of an open-output operation:

$$\circ_j^{\mathfrak{o}} : (\mathsf{P}(m) \otimes \mathsf{Q}(n)) \otimes (\mathsf{P}(m') \otimes \mathsf{Q}(n')) \to \mathsf{P}(m + m') \otimes \mathsf{Q}(n + n' - 1)$$

uses the operad structure of Q and the commutative product $\mathsf{Com} \to \mathsf{P}$.

Algebras over $\mathsf{P} \otimes \mathsf{Q}$ are triplets (B, A, ν) where B is a P-algebra, A is a Q-algebra, and $\nu : B \otimes A \to A$ is an action that makes A into a Q-algebra over the commutative algebra B (cf. ibid. for the definition).

Remark 1.4.23. An Eckmann–Hilton-type argument shows that the algebra structure of P defined by the morphism $\mathsf{Com} \to \mathsf{P}$ has to be commutative for the composition product to even be associative.

Voronov's version of the Swiss-Cheese $\mathsf{SC}^{\mathrm{vor}}$ operad then satisfies:

$$\mathsf{sc}^{\mathrm{vor}} := H_*(\mathsf{SC}^{\mathrm{vor}}) \cong \mathsf{Ger} \otimes \mathsf{Ass}.$$

This isomorphism is moreover an isomorphism of Hopf operads.

In the case of $\mathsf{sc} = H_*(\mathsf{SC})$, one has to use the unital structures of Ass and Ger. We have (and this is still an isomorphism of Hopf operads):

$$\mathsf{sc}_+ := H_*(\mathsf{SC}_+) \cong \mathsf{Ger}_+ \otimes \mathsf{Ass}_+.$$

If we remove the components with zero closed inputs and zero open inputs, we then get $\mathsf{sc} = H_*(\mathsf{SC})$, a relative operad over Ger whose algebras are described above Definition 1.4.22. But one should not forget that:

$$\begin{aligned} \mathsf{sc}(0, m) &= \mathsf{Ger}_+(m) \otimes \mathsf{Ass}_+(0) = \mathsf{Ger}(m) \\ \neq \mathsf{sc}^{\mathrm{vor}}(0, m) &= \mathsf{Ger}(m) \otimes \mathsf{Ass}(0) = 0. \end{aligned}$$

Indeed, we still keep the components that have a nonzero number of total inputs. We have in particular that $\mathsf{sc}(0, 1) \cong \mathsf{Ger}(1) = \mathbb{Q}$ is spanned by the morphism between the Gerstenhaber algebra to the associative algebra. We use the notation:

$$\mathsf{sc} = \mathsf{Ger}_+ \otimes_0 \mathsf{Ass}_+$$

to express the fact that sc is obtained as the Voronov product $\mathsf{Ger}_+ \otimes \mathsf{Ass}_+$ from which we remove the components with zero closed inputs and zero open inputs.

Comparison

By theorems of Kontsevich [Kon99] ($\Bbbk = \mathbb{R}, n \geq 2$) and Tamarkin [Tam03] ($\Bbbk = \mathbb{Q}, n = 2$), the little disks operads are formal, i.e. $C_*(\mathsf{D}_n^+) \simeq H_*(\mathsf{D}_n^+)$. Fresse and Willwacher [FW15] give another proof of this result ($\Bbbk = \mathbb{Q}, n \geq 3$), and show that it can be enhanced in the rational homotopy context: there is a rational equivalence of simplicial operads $\mathtt{D}_n \simeq_{\mathbb{Q}} \langle H^*(\mathtt{D}_n)\rangle^{\mathbb{L}}$, which implies the rational formality of $\mathtt{D}_n$. In low dimensions, we thus have $\mathtt{D}_1 \simeq_{\mathbb{Q}} \langle \mathtt{Ass}^*\rangle^{\mathbb{L}}$ (easy computation). Tamarkin proves that, from the existence of rational Drinfeld associators, it follows that $\mathtt{D}_2 \simeq_{\mathbb{Q}} \langle \mathtt{Ger}^*\rangle^{\mathbb{L}}$. These rational equivalences are also compatible with the unital structures.

Given two (Hopf) cooperads $\mathtt{P}_c, \mathtt{Q}_c$, and a morphism $\mathtt{P}_c \to \mathtt{Com}^*$, one can define the Voronov product $\mathtt{P}_c \otimes \mathtt{Q}_c$, similarly to Definition 1.4.22. This is a relative (Hopf) cooperad under $\mathtt{P}_c$. It is defined by formulas that are formally dual to the ones defining $\mathtt{P} \otimes \mathtt{Q}$. If these cooperads admit counital extensions, we can similarly define $\mathtt{P}_c^+ \otimes_0 \mathtt{Q}_c^+$.

The inclusion $\mathtt{D}_1^+ \hookrightarrow \mathtt{D}_2^+$ induces in cohomology a morphism of Hopf cooperads $\mathtt{Ger}_+^* \to \mathtt{Ass}_+^*$. This morphism factors through a morphism $\mathtt{Ger}_+^* \to \mathtt{Com}_+^*$ (the coproduct of a Gerstenhaber coalgebra is cocommutative). We then get an isomorphism:

$$\mathtt{sc}^* = H^*(\mathtt{SC}) \cong (\mathtt{Ger}_+ \otimes_0 \mathtt{Ass}_+)^* \cong \mathtt{Ger}_+^* \otimes_0 \mathtt{Ass}_+^*.$$

The morphism $\mathtt{Ger}_+^* \to \mathtt{Com}_+^*$ induces a morphism of simplicial operads $\mathtt{Com}_+ \cong \langle \mathtt{Com}_+^*\rangle^{\mathbb{L}} \to \langle \mathtt{Ger}_+^*\rangle^{\mathbb{L}}$. Since the realization functor is monoidal, we get:[4]

$$\langle H^*(\mathtt{SC})\rangle^{\mathbb{L}} \simeq \langle \mathtt{Ger}_+^*\rangle^{\mathbb{L}} \times_0 \langle \mathtt{Ass}_+^*\rangle^{\mathbb{L}}.$$

It is known that $\mathtt{PaP} \simeq \pi\mathtt{D}_1$ and $\widehat{\mathtt{CD}} \simeq_{\mathbb{Q}} \pi\mathtt{D}_2$ [Fre17, §5, §10]. There is an obvious morphism $\mathtt{Com} \to \widehat{\mathtt{CD}}$ sending the generator to the empty chord diagram (see [FW15]), which we can use to build the Voronov product of the operads in groupoids $\widehat{\mathtt{CD}}$ and $\mathtt{PaP}$. Since the fundamental groupoid functor is monoidal too, we finally have:

$$\pi\langle \mathtt{sc}^*\rangle^{\mathbb{L}} \simeq \pi\langle \mathtt{Ger}_+^*\rangle^{\mathbb{L}} \times \pi\langle \mathtt{Ass}_+^*\rangle^{\mathbb{L}} \simeq_{\mathbb{Q}} \widehat{\mathtt{CD}}_+ \times_0 \mathtt{PaP}_+.$$

The operad $\mathtt{SC}$ is not formal [Liv15], therefore $\pi\mathtt{SC}$ is *not* equivalent to

$$\pi\langle \mathtt{sc}^*\rangle^{\mathbb{L}} \simeq_{\mathbb{Q}} \mathtt{PaP}_+ \times_0 \widehat{\mathtt{CD}}_+.$$

Thus, our construction $\mathtt{Pa}\mathtt{P}\widehat{\mathtt{CD}}_+^{\phi}$ rectifies the model arising from the homology of $\mathtt{SC}$ to retrieve, from $\mathtt{PaP}$ and $\widehat{\mathtt{CD}}$, an operad in groupoids which is actually rationally equivalent to $\pi\mathtt{SC}$.

[4]The monoidal structure on simplicial sets being the Cartesian product, we denote the Voronov product of two simplicial operads with $\times$ instead of $\otimes$, and we use $\times_0$ to say that we remove the components with zero inputs.

2 The Lambrechts–Stanley Model of Configuration Spaces

Let M be a closed smooth n-manifold and consider the ordered configuration space of k points in M:

$$\mathrm{Conf}_k(M) := \{(x_1, \dots, x_k) \in M^n \mid x_i \neq x_j \ \forall i \neq j\}.$$

The homotopy type of these spaces is notoriously hard to compute. For example there exist two homotopy equivalent (non simply connected) closed 3-manifolds such that their configuration spaces are not homotopy equivalent [LS05b].

A theorem of Lambrechts–Stanley [LS08b] shows that a simply connected closed manifold M always admits a Poincaré duality model A (in the sense of rational homotopy theory). They build a CDGA $\mathtt{G}_A(k)$ [LS08a] out of A and show that this CDGA is quasi-isomorphic as Σ_k-dg-modules to $A^*_{\mathrm{PL}}(\mathrm{Conf}_k(M))$.

When M is a smooth complex projective variety, Kriz [Kri94] had previously shown that $\mathtt{G}_{H^*(M)}(k)$ is actually a rational CDGA model for $\mathrm{Conf}_k(M)$. The CDGA $\mathtt{G}_{H^*(M)}(k)$ is the $\mathtt{E}^2$ page of a spectral sequence of Cohen–Taylor [CT78] that converges to the cohomology of $\mathrm{Conf}_k(M)$. Totaro [Tot96] has shown that for a smooth complex compact projective variety, the spectral sequence only has one nonzero differential. When $k = 2$, $\mathtt{G}_A(2)$ is known to be a model of $\mathrm{Conf}_2(M)$ either when M is 2-connected [LS04] or when $\dim M$ is even [Cor15]. Lambrechts and Stanley conjecture that $\mathtt{G}_A(k)$ is a rational model of $\mathrm{Conf}_k(M)$ for any simply connected manifold [LS08b].

We prove this conjecture over $\mathbb{R}$ for manifolds of dimension at least 4. This proves as a corollary that the real homotopy type of $\mathrm{Conf}_k(M)$ only depends on the real homotopy type of M and its Poincaré duality.

We use that the space $\mathrm{Conf}_k(M)$ is homotopy equivalent to its Fulton–MacPherson compactification $\mathtt{FM}_M(k)$ [FM94; AS94; Sin04]. When M is framed, these compactifications assemble into a right module $\mathtt{FM}_M$ over the Fulton–MacPherson operad $\mathtt{FM}_n$, an operad weakly equivalent to the little n-disks operad [May72; BV73]. We show that the Lambrechts–Stanley model is compatible with this action of the little disks operad.

Based on "The Lambrechts–Stanley Model of Configuration Spaces", preprint (submitted), arXiv:1608.08054.

The little n-disks operad D_n is formal [Kon99; Tam03; LV14; FW15], i.e. its operad of singular chains $C_*(\mathsf{D}_n)$ is quasi-isomorphic to its homology $\mathsf{e}_n := H_*(\mathsf{D}_n)$. This formality result can be strengthened [LV14; FW15] so that it holds in the category of Hopf cooperads, taking into account the CDGA structures of $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(k))$ and the dual $\mathsf{e}_n^\vee(k)$.

By general arguments, the formality theorem implies that there exists a homotopy class of right e_n-modules e_M such that the pair $(\mathsf{e}_M, \mathsf{e}_n)$ is quasi-isomorphic to the chain complex of the pair $(\mathsf{FM}_M, \mathsf{FM}_n)$. We show that $\mathsf{G}_A = \{\mathsf{G}_A(k)\}_{k\leq 0}$ is a Hopf right $\mathsf{e}_n^\vee$-comodule whose dual is a representative of e_M. Our results are summarized by:

Theorem C (Theorem 2.4.14)**.** *Let M be a smooth simply connected closed n-manifold, where $n \geq 4$. Then for any Poincaré duality model A of M and for all $k \geq 0$, the CDGA $\mathsf{G}_A(k)$ defined by Lambrechts and Stanley is Σ_k-equivariantly weakly equivalent to $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_M(k))$.*

If $\chi(M) = 0$, the collection $\mathsf{G}_A = \{\mathsf{G}_A(k)\}_{k\geq 0}$ moreover forms a Hopf right $\mathsf{e}_n^\vee$-comodule. If M is framed, then $(\mathsf{G}_A, \mathsf{e}_n^\vee)$ is weakly equivalent to $(\Omega^*_{\mathrm{PA}}(\mathsf{FM}_M), \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n))$ as a Hopf right comodule.

As Corollary 2.4.36, we obtain that if two smooth closed simply connected manifolds of dimension at least 4 have the same real homotopy type, then so do their configuration spaces.

In dimension 3, the only simply connected manifold is the 3-sphere, which is framed. The Lambrechts–Stanley conjecture is satisfied over $\mathbb{Q}$ in this case (Proposition 2.4.37), and the collection $\mathsf{G}_{H^*(S^3)}$ is still a Hopf right $\mathsf{e}_3^\vee$-comodule. We conjecture that the model is also compatible with the action of the Fulton–MacPherson operad.

Factorization homology, an invariant of framed n-manifolds defined from an D_n-algebra, may be computed via a derived tensor product over the D_n operad [AF15]. The Taylor tower in the Goowillie–Weiss calculus of embeddings may similarly be computed via a derived Hom [GW99; BW13]. It follows from a result of [Tur13, Section 5.1] that FM_M may be used for this purpose. Therefore our theorem shows that G_A may be used for computing factorization homology or the Taylor tower.

The proof of this theorem is inspired by Kontsevich's proof of the formality theorem, and is radically different from the ideas of the paper [LS08a]. It involves an intermediary Hopf right comodule of labeled graphs Graphs_R. This comodule is similar to a comodule developed by Campos–Willwacher [CW16], which is isomorphic to our construction applied to $R = S(\tilde{H}^*(M))$. Despite this similarity, their whole approach is different, and they manage to prove that $\mathsf{Graphs}_{S(\tilde{H}^*(M))}$ is quasi-isomorphic to $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_M)$ even for non simply connected manifolds. They also prove that in the simply connected case, the real homotopy types of configuration spaces only depend on the real homotopy type of the manifold.

Outline In Section 2.1, we recall some background on the little disks operad, the Fulton–MacPherson compactification, the proof of Kontsevich formality, Poincaré duality CDGAs, and the Lambrechts–Stanley CDGAs. In Section 2.2, we build out of these CDGAs a Hopf right $\mathsf{e}_n^\vee$-comodule G_A. In Section 2.3, inspired by the proof of the formality theorem, we construct the labeled graph complex $\mathtt{Graphs}_R$ which will be used to connect this comodule to $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_M)$. In Section 2.4, we prove that the zigzag of Hopf right comodule morphisms between G_A and $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_M)$ is a weak equivalence. In Section 2.5, we use our model to compute factorization homology of framed manifolds and we compare the result to a complex obtained by Knudsen. Finally, in Section 2.6 we work out a variant of our theorem for the only simply connected surface using the formality of the framed little 2-disks operad, and we present a conjecture about higher dimensional oriented manifolds.

2.1 Background and recollections

We assume basic proficiency with Hopf (co)operads and (co)modules over (co)operads. References include the book [LV12] for (co)operads and the books [Fre09; Fre17] for (co)modules over (co)operads and for Hopf (co)operads.

2.1.1 Conventions

All our dg-modules will have a cohomological grading:

$$V = \bigoplus_{n\in\mathbb{Z}} V^n,$$

and the differentials raise degrees by one: $\deg(dx) = \deg(x) + 1$. If V, W are dg-modules and $v \otimes w \in V \otimes W$, we let $(v \otimes w)^{21} = (-1)^{(\deg v)(\deg w)} w \otimes v$ and we extend this linearly to $V \otimes W$. We will let $V[k]$ be the desuspension, defined by $(V[k])^n = V^{n+k}$. We will call CDGAs the (graded) commutative unital algebras in the category of dg-modules.

Remark 2.1.1. There is a Quillen adjunction between the category of $\mathbb{Z}$-graded CDGAs and the category of $\mathbb{N}$-graded CDGAs. If A is a cofibrant $\mathbb{N}$-graded CDGA which is connected, i.e. satisfying $H^0(A) = \Bbbk$, then it is also cofibrant when seen as a $\mathbb{Z}$-graded CDGA. Therefore if two connected $\mathbb{N}$-graded CDGAs are weakly equivalent (quasi-isomorphic) in the category of $\mathbb{Z}$-graded CDGAs, then they are also weakly equivalent in the category of $\mathbb{N}$-graded CDGAs.

We index our (co)operads by finite sets instead of integers to ease the writing of some formulas. If $W \subset U$ is a subset, we define the quotient $U/W = (U - W) \sqcup \{*\}$, where $*$ represents the class of W; note that $U/\emptyset = U \sqcup \{*\}$. An operad P is given by a functor from the category of finite sets and bijections (also known as a symmetric collection) to the category of dg-modules, a unit $\Bbbk \to \mathsf{P}(\{*\})$, as well as composition maps for every pair of sets:

$$\circ_W : \mathsf{P}(U/W) \otimes \mathsf{P}(W) \to \mathsf{P}(U),$$

satisfying the usual associativity, unity and equivariance conditions.

Dually, a cooperad C is given by a symmetric collection, a counit $\mathsf{C}(\{*\}) \to \Bbbk$, and cocomposition maps for every pair $(U \supset W)$

$$\circ_W^\vee : \mathsf{C}(U) \to \mathsf{C}(U/W) \otimes \mathsf{C}(W).$$

Using the terminology of Fresse [Fre17], we call Hopf cooperads the cooperads in the category of CDGAs.

Remark 2.1.2. The constructions of Fresse [Fre17] are done in the category of $\mathbb{N}$-graded CDGAs. They extend to the setting of $\mathbb{Z}$-graded CDGAs. The previous remark extends to Hopf cooperads.

Let $\underline{k} = \{1, \dots, k\}$. We recover the usual notion of a cooperad indexed by the integers by considering the collection $\{\mathsf{C}(\underline{k})\}_{k \geq 0}$, and the cocomposition maps $\circ_i^\vee : \mathsf{C}(\underline{k+l-1}) \to \mathsf{C}(\underline{k}) \otimes \mathsf{C}(\underline{l})$ correspond to $\circ^\vee_{\{i,\dots,i+l-1\}}$.

We do not generally assume that our (co)operads are trivial in arity zero, but they will satisfy $\mathsf{P}(\emptyset) = \Bbbk$ (resp. $\mathsf{C}(\emptyset) = \Bbbk$). With this assumption we get (co)operad structures which are actually equivalent to the structure of Λ-(co)operads considered by Fresse [Fre17].

We will consider right (co)modules over (co)operads. Given an operad P, a right P-module is a symmetric collection M equipped with composition maps:

$$\circ_W : \mathsf{M}(U/W) \otimes \mathsf{P}(W) \to \mathsf{M}(U)$$

satisfying the usual associativity, unity and equivariance conditions. A right comodule over a cooperad is defined dually. If C is a Hopf cooperad, then a right Hopf C-comodule is a C-comodule N such that all the $\mathsf{N}(U)$ are CDGAs and all the maps $\circ_W^\vee$ are morphisms of CDGAs.

Definition 2.1.3. Let C (resp. C') be a Hopf cooperad and N (resp. N') be a Hopf right comodule over C (resp. C'). A **morphism of Hopf right comodules** $f = (f_{\mathsf{N}}, f_{\mathsf{C}}) : (\mathsf{N}, \mathsf{C}) \to (\mathsf{N}', \mathsf{C}')$ is a pair consisting of a morphism of Hopf cooperads $f_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}'$, and a map of Hopf right C'-comodules $f_{\mathsf{N}} : \mathsf{N} \to \mathsf{N}'$, where N has the C-comodule structure induced by f_{C}. It is said to be a **quasi-isomorphism** if both f_{C} and f_{N} are quasi-isomorphisms of dg-modules in each arity.

If the context is clear, we will allow ourselves to remove the cooperads from the notation in the morphism.

Definition 2.1.4. A Hopf right C-module N is said to be **weakly equivalent** to a Hopf right C'-module N' if the pair (N, C) can be connected to the pair $(\mathsf{N}', \mathsf{C}')$ through a zigzag of quasi-isomorphisms.

2.1.2 Little disks and related objects

The little disks operad D_n is a topological operad initially introduced by May and Boardman–Vogt [May72; BV73] to study iterated loop spaces. Its homology $\mathsf{e}_n := H_*(\mathsf{D}_n)$ is described by a theorem of Cohen [Coh76]: it is either the operad encoding associative algebras for $n = 1$, or the one encoding $(n-1)$-Poisson algebras for $n \geq 2$.

For technical reasons, we instead consider the Fulton–MacPherson operad FM_n, introduced by Fulton–MacPherson [FM94] in the complex context and adapted by Axelrod–Singer [AS94] to the real context. Each space $\mathsf{FM}_n(\underline{k})$ is a compactification of the configuration space $\mathrm{Conf}_k(\mathbb{R}^n)$, where roughly speaking points can become "infinitesimally close". Using insertion of infinitesimal configurations, they assemble to form a topological operad, weakly equivalent to D_n. We refer to [Sin04] for a detailed treatment.

The first two spaces $\mathsf{FM}_n(\emptyset) = \mathsf{FM}_n(\underline{1}) = *$ are singletons, and $\mathsf{FM}_n(\underline{2}) = S^{n-1}$ is a sphere. We will let

$$\mathrm{vol}_{n-1} \in \Omega^{n-1}_{\mathrm{PA}}(S^{n-1}) = \Omega^{n-1}_{\mathrm{PA}}(\mathsf{FM}_n(\underline{2})) \tag{2.1.5}$$

be the top volume form of $\mathsf{FM}_n(\underline{2})$. For $k \geq 2$ and $i \neq j \in \underline{k}$, we also define the projection maps

$$p_{ij} : \mathsf{FM}_n(\underline{k}) \to \mathsf{FM}_n(\underline{2}) \tag{2.1.6}$$

that forget all but two points in the configuration.

To follow Kontsevich's proof of the formality theorem [Kon99], we use the theory of semi-algebraic sets, as developed in [KS00; Har+11]. A semi-algebraic set is a subset of $\mathbb{R}^N$ defined by finite unions of finite intersections of zero sets of polynomials and polynomial inequalities. There is a functor Ω^*_{PA} of "piecewise algebraic differential forms", analogous to the de Rham complex, taking a semi-algebraic set to a real CDGA. If M is a compact semi-algebraic smooth manifold, then $\Omega^*_{\mathrm{PA}}(M) \simeq \Omega^*_{\mathrm{dR}}(M)$.

The spaces $\mathsf{FM}_n(\underline{k})$ are all semi-algebraic stratified manifolds. The dimension of $\mathsf{FM}_n(\underline{k})$ is $nk - n - 1$ for $k \geq 2$, and is 0 otherwise.

The functor Ω^*_{PA} is monoidal, but contravariant; it follows that $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$ is an "almost" Hopf cooperad. It satisfies a slightly modified version of the cooperad axioms coming from the fact that Ω^*_{PA} is not strongly monoidal, as explained in [LV14, Definition 3.1] (they call it a "CDGA model"): the insertion maps $\circ_W$ become zigzags

$$\Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(U)) \xrightarrow{\circ^*_W} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(U/W) \times \mathsf{FM}_n(W)) \xleftarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(U/W)) \otimes \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(W)),$$

where the second map is the Künneth quasi-isomorphism. If C is a Hopf cooperad, an "almost" morphism $f : \mathsf{C} \to \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$ is a collection of CDGA morphisms $f_U : \mathsf{C}(U) \to \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(U))$ for all U, such that the following diagrams commute:

$$\begin{tikzcd}
\mathsf{C}(U) \arrow[r, "f_U"] \arrow[dd, "\circ^\vee_W"] & \Omega^*_{\mathrm{PA}}(FM_n(U)) \arrow[d, "\circ^*_W"] \\
& \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(U/W) \times \mathsf{FM}_n(W)) \\
\mathsf{C}(U/W) \otimes \mathsf{C}(W) \arrow[r, "f_{U/W} \otimes f_W"] & \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(U/W)) \otimes \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(W)) \arrow[u, "\sim"']
\end{tikzcd}$$

We will generally omit the adjective "almost", keeping in mind that some commutative diagrams are a bit more complicated than at first glance.

If M is a manifold, the configuration space $\mathrm{Conf}_k(M)$ can similarly be compactified to give a space $\mathsf{FM}_M(\underline{k})$. When M is framed, these spaces assemble to form a topological right module over FM_n, by inserting infinitesimal configurations. By the Nash–Tognoli Theorem [Nas52; Tog73], any closed smooth manifold is homeomorphic to a semi-algebraic subset of $\mathbb{R}^N$ for some N, and in this way $\mathsf{FM}_M(\underline{k})$ becomes a stratified semi-algebraic manifold of dimension nk.

By the same reasoning as above, if M is framed, then $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_M)$ becomes an "almost" Hopf right comodule over $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$. As before, if N is a Hopf right C-comodule, where C is a cooperad equipped with an "almost" morphism $f : \mathsf{C} \to \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$, then an "almost" morphism $g : \mathsf{N} \to \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M)$ is a collection of CDGA morphisms $g_U : \mathsf{N}(U) \to \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M(U))$ that make the following diagrams commute:

$$\begin{tikzcd}
\mathsf{N}(U) \arrow[r, "f_U"] \arrow[dd, "\circ^\vee_W"] & \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M(U)) \arrow[d, "\circ^*_W"] \\
& \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M(U/W) \times \mathsf{FM}_n(W)) \\
\mathsf{N}(U/W) \otimes \mathsf{C}(W) \arrow[r, "g_{U/W} \otimes f_W"] & \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M(U/W)) \otimes \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n(W)) \arrow[u, "\sim"']
\end{tikzcd}$$

Remark 2.1.7. Following [Fre17, Section II.10.1], there is a construction $\Omega^*_\sharp$ that turns a simplicial operad P into a Hopf cooperad and such that a morphism of

Hopf cooperads $\mathsf{C} \to \Omega^*_\sharp(\mathsf{P})$ is the same thing as an "almost" morphism $\mathsf{C} \to \Omega^*(\mathsf{P})$, where Ω^* is the functor of Sullivan forms. Moreover there is a canonical collection of maps $(\Omega^*_\sharp(\mathsf{P}))(U) \to \Omega^*(\mathsf{P}(U))$, which are weak equivalences if P is a cofibrant operad. Such a functor is built by considering the right adjoint of the functor on operads induced by the Sullivan realization functor, which is monoidal.

Although we will not use it in the rest of the paper, a similar construction can be carried out for Ω^*_{PA}. The proof follows step by step the constructions carried out in [Fre17, Section II.10.1]. The key points are the two facts that for a given $n \geq 0$, the complex $\Omega^*_{\mathrm{PA}}(\Delta^n)$ is acyclic (a.k.a. the Poincaré lemma), and that for a given $k \geq 0$, the simplicial vector space $\Omega^k_{\mathrm{PA}}(\Delta^\bullet)$ is acyclic too. These are respectively [Har+11, Lemma 6.3 and Lemma 6.10] (where the last property is called the "extendability" of $\Omega^*_{\mathrm{PA}}(\Delta^\bullet)$ and is used to prove the Mayer–Vietoris theorem for PA forms). Then one can define a functor $G^{\mathrm{PA}}_\bullet(A) := \mathrm{Hom}_{\mathsf{CDGA}}(A, \Omega^*_{\mathrm{PA}}(\Delta^\bullet))$ on CDGAs, and use the right adjoint of its extension to the category of Hopf cooperads to define and prove the properties of $\Omega^*_{\mathrm{PA},\sharp}$. We can also extend this construction to modules over operads, by similar arguments.

2.1.3 Formality of the little disks operad

Kontsevich's formality theorem [Kon99; LV14] can be summarized by the fact that $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)$ is weakly equivalent to $\mathsf{e}^\vee_n$ as a Hopf cooperads. We outline here the proof as we will mimic its pattern for our theorem. We will use the formalism of (co)operadic twisting [Wil14; DW15], and we refer to [LV14] for proofs of most of the claims of this section.

The cohomology of D_n The classical description due to Arnold and Cohen of the cohomology $\mathsf{e}^\vee_n(U) = H^*(\mathsf{D}_n(U))$ is:

$$\mathsf{e}^\vee_n(U) = S(\omega_{uv})_{u \neq v \in U}/I \qquad (2.1.8)$$

where $S(-)$ is the free graded commutative algebra, the generators ω_{uv} have cohomological degree $n-1$, and the ideal I is generated by the relations:

$$\omega_{vu} = (-1)^n \omega_{uv}; \ \omega_{uv}^2 = 0; \ \omega_{uv}\omega_{vw} + \omega_{vw}\omega_{wu} + \omega_{wu}\omega_{uv} = 0. \qquad (2.1.9)$$

The cooperadic structure maps are given on generators by:

$$\circ_W^\vee : \mathsf{e}_n^\vee(U) \to \mathsf{e}_n^\vee(U/W) \otimes \mathsf{e}_n^\vee(W)$$
$$\omega_{uv} \mapsto \begin{cases} 1 \otimes \omega_{uv}, & \text{if } u, v \in W; \\ \omega_{*v} \otimes 1, & \text{if } u \in W \text{ and } v \notin W; \\ \omega_{u*} \otimes 1, & \text{if } u \notin W \text{ and } v \in W; \\ \omega_{uv} \otimes 1, & \text{if } u, v \notin W. \end{cases} \tag{2.1.10}$$

Graphs with only external vertices The intermediary cooperad of graphs is built in several steps. In the first step, define a cooperad of graphs with only external vertices, with generators e_{uv} of degree $n-1$:

$$\mathtt{Gra}_n(U) = (S(e_{uv})_{u,v \in A} / (e_{uv}^2 = e_{uu} = 0, e_{vu} = (-1)^n e_{uv}), d = 0). \tag{2.1.11}$$

The CDGA $\mathtt{Gra}_n(U)$ is spanned by words of the type $e_{u_1v_1} \dots e_{u_rv_r}$. Such a word can be viewed as a graph with U as the set of vertices, and an edge between u_i and v_i for each factor $e_{u_iv_i}$. For example, the word e_{uv} is a graph with a single edge between the vertices u and v (see Figure 2.1.1 for another example). Graphs with double edges or edges between a vertex and itself are set to zero. Given such a graph, its set of edges $E_\Gamma \subset \binom{U}{2}$ is well-defined. The vertices of these graphs are called "external", in contrast with the internal vertices that are going to appear in the next part.

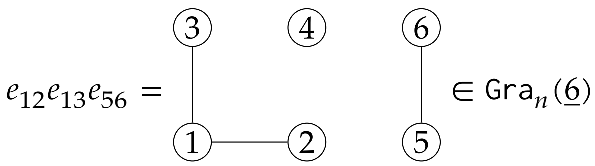


Figure 2.1.1: Example of the correspondence between graphs and words

The Hopf product map $\mathtt{Gra}_n(U) \otimes \mathtt{Gra}_n(U) \to \mathtt{Gra}_n(U)$, from this point of view, consists of gluing two graphs along their vertices. The cooperadic structure map $\circ_W^\vee : \mathtt{Gra}_n(U) \to \mathtt{Gra}_n(U/W) \otimes \mathtt{Gra}_n(W)$ maps a graph Γ to $\pm\Gamma_{U/W} \otimes \Gamma_W$ such that Γ_W is the full subgraph of Γ with vertices W and $\Gamma_{U/W}$ collapses this full subgraph to a single vertex.

On the generators, the formula for $\circ_W^\vee$ is in fact identical to Equation (2.1.10), replacing $\omega_{??}$ by $e_{??}$. This implies that the cooperad $\mathtt{Gra}_n$ maps to $\mathsf{e}_n^\vee$ by sending e_{uv} to ω_{uv}.

The cooperad $\mathtt{Gra}_n$ also maps to $\Omega^*_{\mathrm{PA}}(\mathtt{FM}_n)$ using a map given on generators by:

$$\begin{aligned}\omega' : \mathtt{Gra}_n(U) &\to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n(U))\\ \Gamma &\mapsto \bigwedge_{(u,v)\in E_\Gamma} p^*_{uv}(\mathrm{vol}_{n-1}),\end{aligned} \tag{2.1.12}$$

where vol_{n-1} is the volume form of $\mathtt{FM}_n(2) \cong S^{n-1}$ (2.1.5).

Twisting The second step of the construction is cooperadic twisting, for which our general reference is the appendix of [Wil16] (see also [DW15]). Let $\mathtt{hoLie}_k$ be the operad controlling homotopy Lie algebras shifted by $k+1$ (with $\mathtt{Lie}_k$ being the operad controlling shifted Lie algebras). Let $\mathtt{C}$ be a cooperad, finite-dimensional in every arity and equipped with a map to the dual of $\mathtt{hoLie}_k$. This map can equivalently be seen as a Maurer–Cartan element μ in the deformation complex $\mathrm{Def}(\mathtt{hoLie}_k \to \mathtt{C}^\vee)$, a convolution dg-Lie algebra. Then define:

$$\mathrm{Tw}\,\mathtt{C}(U) := \bigoplus_{i\geq 0}(\mathtt{C}(U \sqcup \underline{i}) \otimes \mathbb{R}[k]^{\otimes i})_{\Sigma_i}.$$

The symmetric module $\mathrm{Tw}\,\mathtt{C}$ can be given a cooperad structure induced by $\mathtt{C}$. Its differential is the sum of the internal differential of $\mathtt{C}$ and a differential coming from the action of μ, acting on both sides of $\mathtt{C}$. Roughly speaking, $\mathrm{Tw}\,\mathtt{C}$ encodes coalgebras over $\mathtt{C}$ with a differential twisted by a “Maurer–Cartan element”.[1] There is also an obvious inclusion $\mathtt{C} \to \mathrm{Tw}\,\mathtt{C}$ that commutes with differentials and cooperad maps. We refer to [Wil16] for the details – one needs to formally dualize the appendix to twist cooperads instead of operads.

If $\mathtt{C}$ satisfies $\mathtt{C}(\emptyset) = \Bbbk$ and is a Hopf cooperad, then $\mathrm{Tw}\,\mathtt{C}$ inherits a Hopf cooperad structure. To multiply an element of $\mathtt{C}(U \sqcup I) \subset \mathrm{Tw}\,\mathtt{C}(U)$ with an element of $\mathtt{C}(U \sqcup J) \subset \mathrm{Tw}\,\mathtt{C}(U)$, first use the maps

$$\mathtt{C}(V) \xrightarrow{\circ^\vee_\emptyset} \mathtt{C}(V/\emptyset) \otimes \mathtt{C}(\emptyset) \cong \mathtt{C}(V \sqcup \{*\})$$

several times to map both elements to $\mathtt{C}(U \sqcup I \sqcup J)$, and then use the CDGA structure of $\mathtt{C}(U \sqcup I \sqcup J)$ to multiply them.

We now turn our attention to graphs. The Hopf cooperad $\mathtt{Gra}_n$ maps into $\mathtt{Lie}^\vee_n$ as follows. The cooperad $\mathtt{Lie}^\vee_n$ is cogenerated by $\mathtt{Lie}^\vee_n(\underline{2})$, and on cogenerators the cooperad map is given by sending $e_{12} \in \mathtt{Gra}_n(\underline{2})$ to the cobracket in $\mathtt{Lie}^\vee_n(\underline{2})$ and all the other graphs to zero. This map to $\mathtt{Lie}^\vee_n$ yields a map to $\mathtt{hoLie}^\vee_n$ by composition with the canonical map $\mathtt{Lie}^\vee_n \xrightarrow{\sim} \mathtt{hoLie}^\vee_n$. Roughly speaking, the Maurer–Cartan element μ is given by Figure 2.1.2.

[1] A coalgebra over $\mathtt{C}$ is automatically a $\mathtt{hoLie}^\vee_k$-coalgebra due to the fixed morphism $\mathtt{C} \to \mathtt{hoLie}^\vee_k$, therefore the notion of Maurer–Cartan element in a $\mathtt{C}$-coalgebra is well-defined.

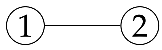


Figure 2.1.2: The Maurer–Cartan element $\mu = e_{12}^{\vee} \in \mathtt{Gra}_n^{\vee}(2)$

The cooperad $\mathtt{Gra}_n$ satisfies $\mathtt{Gra}_n(\emptyset) = \mathbb{R}$. The induced maps $\mathtt{Gra}_n(U) \to \mathtt{Gra}_n(U')$ (for $U \subset U'$) add new vertices with no incident edges. Thus the general framework makes $\operatorname{Tw}\mathtt{Gra}_n$ into a Hopf cooperad, which we now explicitly describe.

The dg-module $\operatorname{Tw}\mathtt{Gra}_n(U)$ is spanned by graphs with two types of vertices: external vertices, which correspond to elements of U, and indistinguishable internal vertices (usually drawn in black). The degree of an edge is $n-1$, and the degree of an internal vertex is $-n$. The differential sends a graph Γ to the sum:

$$d\Gamma = \sum_{\substack{e \in E_\Gamma \\ \text{contractible}}} \pm \Gamma/e,$$

where Γ/e is Γ with the edge e collapsed and e ranges over all "contractible" edges, i.e. edges connecting an internal vertex to another vertex of either kind. See Figure 2.1.3 for an example.

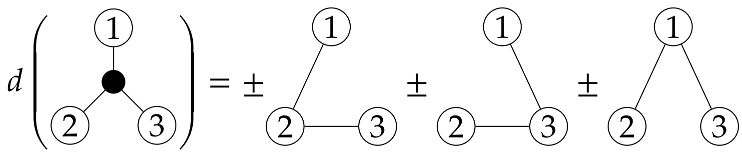


Figure 2.1.3: The differential of $\operatorname{Tw}\mathtt{Gra}_n$

Remark 2.1.13. An edge connected to a univalent internal vertex is not considered contractible: the Maurer–Cartan element $\mu \in \operatorname{Def}(\mathtt{C} \to \mathtt{hoLie}_k^{\vee})$ "acts" on both sides of $\operatorname{Tw}\mathtt{C}$ in the definition of the differential, and for such edges the two contributions cancel out. Such vertices are called **dead ends** by Lambrechts–Volić [LV14].[2]

The product of two graphs glues them along their external vertices only (which is the same thing as adding disjoint internal vertices to both graphs and gluing along all vertices).

[2] Their definition is slightly different, but since we forbid multiple edges and loops, the two definitions are equivalent.

The two morphisms $\mathsf{e}_n^\vee \leftarrow \mathtt{Gra}_n \xrightarrow{\omega'} \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n)$ extend along the inclusion $\mathtt{Gra}_n \subset \mathrm{Tw}\,\mathtt{Gra}_n$ as follows. The extended morphism $\mathrm{Tw}\,\mathtt{Gra}_n \to \mathsf{e}_n^\vee$ simply sends a graph with internal vertices to zero; since dead ends are not contractible, this commutes with differentials. The extended morphism $\omega : \mathrm{Tw}\,\mathtt{Gra}_n \to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n)$ sends a graph $\Gamma \in \mathtt{Gra}_n(U \sqcup I) \subset \mathrm{Tw}\,\mathtt{Gra}_n(U)$ to:

$$\omega(\Gamma) := (p_U)_*(\omega'(\Gamma)) = \int_{\mathtt{FM}_n(U \sqcup I) \to \mathtt{FM}_n(U)} \omega'(\Gamma), \tag{2.1.14}$$

where p_U is the projection that forgets the points of the configuration corresponding to I, and the integral is an integral along the fiber of this semi-algebraic bundle.

Reduction of the graph complex The cooperad $\mathrm{Tw}\,\mathtt{Gra}_n$ does not have the right homotopy type yet. It is reduced by quotienting out all the graphs with connected components consisting exclusively of internal vertices. This is a Hopf cooperad bi-ideal and thus the resulting quotient $\mathtt{Graphs}_n$ is still a Hopf cooperad. One checks that the two morphisms $\mathsf{e}_n^\vee \leftarrow \mathrm{Tw}\,\mathtt{Gra}_n \to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n)$ factor through the quotient (the first one because dead ends are not contractible, the second one because ω vanishes on graphs with only internal vertices). The resulting zigzag:

$$\mathsf{e}_n^\vee \leftarrow \mathtt{Graphs}_n \to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n) \tag{2.1.15}$$

is then a zigzag of weak equivalence of Hopf cooperads by the work of Kontsevich [Kon99], which proves the formality theorem.

2.1.4 Poincaré duality CDGA models

The model for $\Omega^*_{\mathrm{PA}}(\mathtt{FM}_M)$ relies on a Poincaré duality model of M. We mostly borrow the terminology and notations from [LS08b]. Recall that we only consider simply connected manifolds, which are necessarily orientable and thus satisfy Poincaré duality.

Fix an integer n and let A be a connected CDGA (i.e. $A = \Bbbk \oplus A^{\geq 1}$). A Poincaré duality pairing on A is a dg-form $\varepsilon : A \to \Bbbk[-n]$ (i.e. a linear map $A^n \to \Bbbk$ with $\varepsilon \circ d = 0$) such that the induced pairing

$$\begin{aligned} A^k \otimes A^{n-k} &\to \Bbbk \\ a \otimes b &\mapsto \varepsilon(ab) \end{aligned} \tag{2.1.16}$$

is non-degenerate for all k. This implies that $A = A^{\leq n}$, and that $\varepsilon : A^n \to \Bbbk$ is an isomorphism. The pair (A, ε) is called a **Poincaré duality algebra**.

If A is such a Poincaré duality CDGA, then so is its cohomology. The theorem of Lambrechts–Stanley [LS08b] implies that if Ω is a CDGA (over any field) whose minimal model is 1-connected and of finite type,[3] and if $\int : \Omega \to \Bbbk[-n]$ is a dg-form inducing a Poincaré duality structure of dimension n on $H^*(\Omega)$, then Ω is quasi-isomorphic to a Poincaré duality algebra (A, ε) of the same dimension through a zigzag of CDGA morphisms which respect the dg-forms:

$$\begin{tikzcd} \Omega \arrow[dr, "\int"'] & \cdot \arrow[l, "\sim"'] \arrow[r, "\sim"] & \exists A \arrow[dl, "\exists\varepsilon"] \\ & \Bbbk & \end{tikzcd}$$

Remark 2.1.17. Even though it is not written in the statement of the theorem in [LS08b], studying of the proof shows that the diagram above does commute.

Let A be a Poincaré duality CDGA of finite type and let $\{a_i\}$ be a graded basis of A. Consider the dual basis $\{a_i^\vee\}$ with respect to the duality pairing, i.e. $\varepsilon(a_i a_j^\vee) = \delta_{ij}$ is given by the Kronecker symbol. Then the **diagonal cocycle** is defined by the following formula and is independent of the chosen graded basis (see e.g. [FOT08]):

$$\Delta_A = \sum_i (-1)^{|a_i|} a_i \otimes a_i^\vee \in A \otimes A. \tag{2.1.18}$$

The element Δ_A is a cocycle of degree n (this follows from $\varepsilon \circ d = 0$). It satisfies $\Delta^{21} = (-1)^n \Delta$, and for all $a \in A$ it satisfies the equation $(a \otimes 1)\Delta_A = (1 \otimes a)\Delta_A$. Finally, the product $\mu_A : A \otimes A \to A$ sends Δ_A to $\chi(A) \cdot \mathrm{vol}_A$, where $\chi(A)$ is the Euler characteristic of A and $\mathrm{vol}_A \in A^n$ is the preimage of $1 \in \Bbbk$ by $\varepsilon : A^n \to \Bbbk$.

2.1.5 The Lambrechts–Stanley CDGAs

We will give the definition of the CDGA $\mathtt{G}_A(k)$, constructed in [LS08a] in the general case of a Poincaré duality CDGA (see the introduction for a more detailed history).

Let A be a Poincaré duality CDGA of dimension n and let k be an integer. For $1 \le i \ne j \le k$, let $\iota_i : A \to A^{\otimes k}$ be defined by $\iota_i(a) = 1^{\otimes i-1} \otimes a \otimes 1^{\otimes k-i-1}$, and let $\iota_{ij} : A \otimes A \to A^{\otimes k}$ be given by $\iota_{ij}(a \otimes b) = \iota_i(a) \cdot \iota_j(b)$.

Recalling the description of $\mathsf{e}_n^\vee$ in Equation (2.1.8), the CDGA $\mathtt{G}_A(k)$ is then defined by:

$$\mathtt{G}_A(k) = (A^{\otimes k} \otimes \mathsf{e}_n^\vee(k) / (\iota_i(a) \cdot \omega_{ij} = \iota_j(a) \cdot \omega_{ij}), d\omega_{ij} = \iota_{ij}(\Delta_A)). \tag{2.1.19}$$

[3] A chain complex is of "finite type" if it is finite dimensional in each degree. Looking closely at the proof of the theorem, we see that Ω does not need to be of finite type, only its minimal model does.

We will call these CDGAs the Lambrechts–Stanley CDGAs, or **LS CDGAs** for short. For example $\mathsf{G}_A(0) = \Bbbk$, $\mathsf{G}_A(1) = A$, and $\mathsf{G}_A(2)$ is isomorphic to:

$$\mathsf{G}_A(2) \cong \big((A \otimes A) \oplus (A \otimes \omega_{12}), d(a \otimes \omega_{12}) = (a \otimes 1) \cdot \Delta_A = (1 \otimes a) \cdot \Delta_A\big).$$

When M is any simply connected closed manifold, a theorem of Lambrechts and Stanley [LS08a] implies that there exists a Poincaré duality CDGA A which is a rational model for M and such that

$$H^*(\mathsf{G}_A(k); \mathbb{Q}) \cong H^*(\mathsf{FM}_M(k); \mathbb{Q}) \text{ as graded vector spaces.} \tag{2.1.20}$$

2.2 The Hopf right comodule model G_A

In this section we describe the Hopf right $\mathsf{e}_n^\vee$-comodule derived from the LS CDGAs of Section 2.1.5. From now on we work over $\mathbb{R}$, and we fix a simply connected smooth closed manifold M of dimension at least 4. Following Section 2.1.2, we endow M with a fixed semi-algebraic structure.

For now we fix an arbitrary Poincaré duality CDGA model A of M; we will choose one in the next section. We then define the right comodule structure of G_A as follows, using the cooperad structure of $\mathsf{e}_n^\vee$ given by Equation (2.1.10):

Proposition 2.2.1. *If $\chi(M) = 0$, then the following maps go through the quotients defining $\mathsf{G}_A = \{\mathsf{G}_A(k)\}_{k \geq 0}$ and endow it with a Hopf right $\mathsf{e}_n^\vee$-comodule structure:*

$$\begin{aligned} \circ_W^\vee : A^{\otimes U} \otimes \mathsf{e}_n^\vee(U) &\to (A^{\otimes (U/W)} \otimes \mathsf{e}_n^\vee(U/W)) \otimes \mathsf{e}_n^\vee(W) \\ (a_u)_{u \in U} \otimes \omega &\mapsto \underbrace{\big((a_u)_{u \in U - W} \otimes \prod_{w \in W} a_w\big)}_{\in A^{\otimes (U/W)}} \otimes \underbrace{\circ_W^\vee(\omega)}_{\in \mathsf{e}_n^\vee(U/W) \otimes \mathsf{e}_n^\vee(W)} \end{aligned} \tag{2.2.2}$$

In informal terms, $\circ_W^\vee$ multiplies together all the elements of A indexed by W on the $A^{\otimes U}$ factor and indexes the result by $* \in U/W$, and it applies the cooperadic structure map of $\mathsf{e}_n^\vee$ on the other factor. Note that if $W = \emptyset$, then $\circ_W^\vee$ adds a factor of 1_A (the empty product) indexed by $* \in U/\emptyset = U \sqcup \{*\}$.

We split the proof in three parts: compatibility of the maps with the cooperadic structure of $\mathsf{e}_n^\vee$, factorization of the maps through the quotient, and compatibility with the differential.

Proposition 2.2.1, part 1: Factorization through the quotient. Since the algebra A is commutative, the maps of the proposition commute with multiplication. The ideals defining $\mathsf{G}_A(U)$ are multiplicative ideals, hence it suffices to show that the maps (2.2.2) take the generators $(\iota_u(a) - \iota_v(a)) \cdot \omega_{uv}$ of the ideal to elements of the ideal in the target. We simply check each case:

- If $u, v \in W$, then $\circ_W^\vee(\iota_u(a)\omega_{uv}) = \iota_*(a) \otimes \omega_{uv}$, which is equal to $\circ_W^\vee(\iota_v(a)\omega_{uv})$.
- If $u \in W$ and $v \notin W$, then
$$\circ_W^\vee(\iota_u(a)\omega_{uv}) = \iota_*(a)\omega_{*v} \otimes 1 \equiv \circ_W^\vee(\iota_v(a)\omega_{uv})$$
are congruent modulo the ideal, and the case $u \notin W, v \in W$ is symmetric.
- Finally if $u, v \notin W$ then
$$\circ_W^\vee(\iota_u(a)\omega_{uv}) = \iota_u(a)\omega_{uv} \otimes 1 \equiv \circ_W^\vee(\iota_v(a)\omega_{uv}). \qquad \square$$

Proposition 2.2.1, part 2: Compatibility with the differential. All the maps involved are respectively morphisms of algebras or derivations of algebras, thus it suffices to check the compatibility on the generators $\iota_u(a)$ and ω_{uv}.

The equality $\circ_W^\vee(d(\iota_u(a))) = d(\circ_W^\vee(\iota_u(a)))$ is immediate. For ω_{uv} we again check the three cases. Recall that since our manifold has vanishing Euler characteristic, $\mu_A(\Delta_A) = 0$.

- If $u, v \in W$, then
$$\circ_W^\vee(d\omega_{uv}) = \iota_*(\mu_A(\Delta_A)) = 0,$$
whereas $\circ_W^\vee(\omega_{uv}) = 1 \otimes \omega_{uv}$ and thus $d(\circ_W^\vee(\omega_{uv})) = 0$ (since $d_{\mathsf{e}_n^\vee} = 0$).
- If $u \notin W$ and $v \in W$, then
$$\circ_W^\vee(d\omega_{uv}) = \iota_{*v}(\Delta_A) \otimes 1 = d(\omega_{*v} \otimes 1) = d(\circ_W^\vee(\omega_{uv})),$$
and the case $u \in W, v \notin W$ is symmetric.
- Finally if $u, v \notin W$, then
$$\circ_W^\vee(d\omega_{uv}) = \iota_{uv}(\Delta_A) \otimes 1 = d(\omega_{uv} \otimes 1) = d(\circ_W^\vee(\omega_{uv})). \qquad \square$$

Proposition 2.2.1, part 3: Comodule structure. Although the fact that the cocomposition maps are compatible with the coproduct of $\mathsf{e}_n^\vee$ can easily be proved "by hand", it also follows from general arguments.

Let $\mathtt{Com}^\vee$ be the cooperad of counital cocommutative coalgebras, which is a right comodule over itself, and view A as an operad concentrated in arity 1. We apply the result of the next lemma (Lemma 2.2.3) to $\mathtt{C} = \mathtt{Com}^\vee$. We get that the collection $\mathtt{Com}^\vee \circ A = \{A^{\otimes k}\}_{k \geq 0}$ forms a $\mathtt{Com}^\vee$-comodule. Then the arity-wise tensor product (see [LV12, Section 5.1.12], where this operation is called the Hadamard product):
$$(\mathtt{Com}^\vee \circ A) \boxtimes \mathsf{e}_n^\vee = \{A^{\otimes k} \otimes \mathsf{e}_n^\vee(k)\}_{k \geq 0}$$

is the arity-wise tensor product of a $\mathtt{Com}^\vee$-comodule and an $\mathtt{e}_n^\vee$-comodule, hence it is a $(\mathtt{Com}^\vee \boxtimes \mathtt{e}_n^\vee)$-comodule. The cooperad $\mathtt{Com}^\vee$ is the unit of the arity-wise tensor product of cooperads, hence the result is an $\mathtt{e}_n^\vee$-comodule. It remains to make the easy check that the resulting comodule maps are given by Equation (2.2.2). □

The next very general lemma can for example be found in [CW16, Section 5.2]. Let $\mathtt{C}$ be a cooperad, and see the CDGA A as an operad concentrated in arity 1. Then the commutativity of A implies the existence of a distributive law $t : \mathtt{C} \circ A \to A \circ \mathtt{C}$, given in each arity by:

$$t : (\mathtt{C} \circ A)(\underline{n}) = \mathtt{C}(\underline{n}) \otimes A^{\otimes n} \to (A \circ \mathtt{C})(\underline{n}) = A \otimes \mathtt{C}(\underline{n})$$
$$x \otimes a_1 \otimes \dots \otimes a_n \mapsto a_1 \dots a_n \otimes x$$

Lemma 2.2.3. *Let $\mathtt{N}$ be a right $\mathtt{C}$-comodule, and see A as a symmetric collection concentrated in arity 1. Then $\mathtt{N} \circ A$ is a right $\mathtt{C}$-comodule through the map:*

$$\mathtt{N} \circ A \xrightarrow{\Delta_{\mathtt{N}} \circ 1} \mathtt{N} \circ \mathtt{C} \circ A \xrightarrow{1 \circ t} \mathtt{N} \circ A \circ \mathtt{C}. \quad \square$$

2.3 Labeled graph complexes

In this section we construct the intermediary comodule used to prove the theorem. We will construct a zigzag of CDGAs of the form:

$$\mathtt{G}_A \leftarrow \mathtt{Graphs}_R \to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M).$$

The construction of $\mathtt{Graphs}_R$ follows the same pattern as the one of $\mathtt{Graphs}_n$ from Section 2.1.3, using labeled graphs. If $\chi(M) = 0$, then the collections $\mathtt{G}_A$ and $\mathtt{Graphs}_R$ are Hopf right comodules respectively over $\mathtt{e}_n^\vee$ and a $\mathtt{Graphs}_n$, and the left arrow is a morphism of comodules between $(\mathtt{G}_A, \mathtt{e}_n^\vee)$ and $(\mathtt{Graphs}_R, \mathtt{Graphs}_n)$. When M is moreover framed, $\Omega^*_{\mathrm{PA}}(\mathtt{FM}_M)$ becomes a Hopf right comodule over $\Omega^*_{\mathrm{PA}}(\mathtt{FM}_n)$, and the right arrow is a morphism of comodules.

2.3.1 Graphs with loops and multiple edges

We first define a variant $\mathtt{Graphs}_n^\circlearrowleft$ of $\mathtt{Graphs}_n$, where graphs are allowed to have "loops" (also sometimes known as "tadpoles") and multiple edges. For a finite set U, the CDGA $\mathtt{Gra}_n^\circlearrowleft(U)$ is given by:

$$\mathtt{Gra}_n^\circlearrowleft(U) = (S(e_{uv})_{u,v \in A} / (e_{vu} = (-1)^n e_{uv}), d = 0).$$

The difference with Equation (2.1.11) is that we no longer set $e_{uu} = e_{uv}^2 = 0$.

Remark 2.3.1. When n is even, $e_{uv}^2 = 0$ since $\deg e_{uv} = n - 1$ is odd; and when n is odd, $e_{uu} = (-1)^n e_{uu} = -e_{uu} \implies e_{uu} = 0$.

Like $\mathtt{Gra}_n$, this defines a Hopf cooperad with cocomposition given by an equation similar to Equation (2.1.10):

$$\begin{aligned} \circ_W^\vee : \mathtt{Gra}_n^{\circlearrowleft}(U) &\to \mathtt{Gra}_n^{\circlearrowleft}(U/W) \otimes \mathtt{Gra}_n^{\circlearrowleft}(W) \\ e_{uv} &\mapsto \begin{cases} e_{**} \otimes 1 + 1 \otimes e_{uv}, & \text{if } u, v \in W; \\ e_{*v} \otimes 1, & \text{if } u \in W \text{ and } v \notin W; \\ e_{u*} \otimes 1, & \text{if } u \notin W \text{ and } v \in W; \\ e_{uv} \otimes 1, & \text{if } u, v \notin W. \end{cases} \end{aligned} \tag{2.3.2}$$

This new cooperad has a graphical description similar to $\mathtt{Gra}_n$. The difference in the cooperad structure is that when we collapse a subgraph, we sum over all ways of choosing whether edges are in the subgraph or not; if they aren't, then they yield a loop. See Figure 2.3.1 for an example. The cooperad $\mathtt{Gra}_n$ is the quotient of $\mathtt{Gra}_n^{\circlearrowleft}$ by the ideal generated by the loops and the multiple edges.

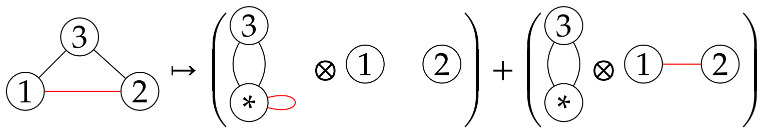


Figure 2.3.1: Example of cocomposition in $\mathtt{Gra}_n^{\circlearrowleft}$

The element $e_{12}^\vee \in (\mathtt{Gra}_n^{\circlearrowleft})^\vee(\underline{2})$ still defines a Maurer–Cartan element

$$\mu := e_{12}^\vee \in \mathrm{Def}(\mathtt{hoLie}_n \to (\mathtt{Gra}_n^{\circlearrowleft})^\vee),$$

which allows us to define the twisted Hopf cooperad Tw $\mathtt{Gra}_n^{\circlearrowleft}$. It has a graphical description similar to Tw $\mathtt{Gra}_n$ with internal and external vertices. Finally we can quotient by graphs containing connected component consisting exclusively of internal vertices to get a Hopf cooperad $\mathtt{Graphs}_n^{\circlearrowleft}$.

2.3.2 External vertices: $\mathtt{Gra}_R$

We construct a collection of CDGAs $\mathtt{Gra}_R$, corresponding to the first step in the construction of $\mathtt{Graphs}_n$ of Section 2.1.3. We first apply the formalism of Section 2.1.4 to $\Omega^*_{\mathrm{PA}}(M)$ in order to obtain a Poincaré duality CDGA out of M:

Theorem 2.3.3 (Lambrechts–Stanley [LS08b])**.** *There exists a zigzag of weak equivalences of CDGAs*

$$A \xleftarrow{\rho} R \xrightarrow{\sigma} \Omega^*_{\mathrm{PA}}(M),$$

such that:

1. *A is a Poincaré duality CDGA of dimension n;*
2. *R is a quasi-free CDGA generated in degrees ≥ 2;*
3. *For all $x \in R$, $\varepsilon_A(\rho(x)) = \int_M \sigma(x)$ (see Remark 2.1.17.*

If $\chi(M) = 0$ then the diagonal cocycle of any Poincaré duality model A of M satisfies $\mu_A(\Delta_A) = 0$. We will require the following technical lemma.

Proposition 2.3.4. *One can choose the zigzag of Theorem 2.3.3 such there exists a symmetric cocycle $\Delta_R \in R \otimes R$ of degree n satisfying $(\rho \otimes \rho)(\Delta_R) = \Delta_A$. If $\chi(M) = 0$ we can moreover choose it so that $\mu_R(\Delta_R) = 0$.*

We follow closely the proof of [LS08b] to obtain the result.

Proof (case $n \leq 6$). When $n \leq 6$, the CDGA $\Omega^*_{\mathrm{PA}}(M)$ is formal [NM78, Proposition 4.6]. We choose $A = (H^*(M), d_A = 0)$, and R to be the minimal model of M.

By Künneth's formula, $R \otimes R \to A \otimes A$ is a quasi-isomorphism. Since $d\Delta_A = 0$, there exists some cocycle $\Delta' \in R \otimes R$ such that $\rho(\Delta') = \Delta_A + d\alpha = \Delta_A$ (since $d_A = 0$).

Let us now assume that $\chi(M) = 0$. Then $\rho(\mu_R(\Delta')) = \mu_A(\Delta_A) = \chi(A)\mathrm{vol}_A = 0$. Since ρ is a quasi-isomorphism and hence injective in cohomology, $\mu_R(\Delta') = d\beta$ for some $\beta \in R \otimes R$. We now let $\Delta'' = \Delta' - d\beta \otimes 1$, so that $\mu_R(\Delta'') = 0$, and

$$\rho(\Delta'') = \rho(\Delta') - \rho(d\beta) \otimes 1 = \Delta_A.$$

If $\chi(M) \neq 0$ we simply let $\Delta'' = \Delta'$. We finally set $\Delta_R = \frac{1}{2}(\Delta'' + (-1^n)(\Delta'')^{21})$, which is symmetric and still satisfies all the required properties. □

Proof (case $n \geq 7$). When $n \geq 7$, the proof of Lambrechts and Stanley builds a zigzag of weak equivalences:

$$A \xleftarrow{\rho} R \leftarrow R' \to \Omega^*_{\mathrm{PA}}(M),$$

where R' is the minimal model of M, the CDGA R is obtained from R' by successively adjoining cells of degree $\geq n/2 + 1$, and the Poincaré duality CDGA A is a quotient of R by an ideal of "orphans". By construction, this zigzag is compatible with ε_A and $\int_M$.

The minimal model R' is quasi-free, and since M is simply connected it is generated in degrees ≥ 2. The CDGA R is obtained from R' by a cofibrant cellular extension, adjoining cells of degree greater than 2. It follows that R is cofibrant and quasi-free generated in degrees ≥ 2. Since $R' \to R$ is an acyclic cofibration and $\varepsilon : R' \to \mathbb{R}$ is a fibration, we can invert $R' \to R$ up to homotopy while preserving ε. Composing with $R' \to \Omega^*_{\mathrm{PA}}(M)$ yields a morphism $\sigma : R \to \Omega^*_{\mathrm{PA}}(M)$ that still satisfies $\varepsilon_A \circ \rho = \int_M \sigma(-)$, and we therefore get a zigzag $A \leftarrow R \to \Omega^*_{\mathrm{PA}}(M)$.

The morphism ρ is a quasi-isomorphism, so there exists some cocycle $\tilde{\Delta} \in R \otimes R$ such that $\rho(\tilde{\Delta}) = \Delta_A + d\alpha$ for some α. By surjectivity of ρ (it is a quotient map) there is some β such that $\rho(\beta) = \alpha$; we let $\Delta' = \tilde{\Delta} - d\beta$, and now $\rho(\Delta') = \Delta_A$.

Let us assume for the moment that $\chi(M) = 0$. Then the cocycle $\mu_R(\Delta') \in R$ satisfies $\rho(\mu_R(\Delta')) = \mu_A(\Delta_A) = 0$, i.e. it is in the kernel of ρ. It follows that the cocycle $\Delta'' = \Delta' - \mu_R(\Delta') \otimes 1$ is still mapped to Δ_A by ρ, and satisfies $\mu_R(\Delta'') = 0$.

If $\chi(M) \neq 0$ we just let $\Delta'' = \Delta'$. Finally we symmetrize Δ'' to get the Δ_R of the lemma, which satisfies all the requirements. □

From now on we keep the zigzag and the element Δ_R of the previous lemma fixed until the end of Section 2.4.

Definition 2.3.5. The CDGA of R**-labeled graphs with loops** on the set U is given by:

$$\mathtt{Gra}_R^{\circlearrowleft}(U) = (R^{\otimes U} \otimes \mathtt{Gra}_n^{\circlearrowleft}(U), de_{uv} = \iota_{uv}(\Delta_R)).$$

Remark 2.3.6. It follows from the definition that $de_{uu} = \iota_{uu}(\Delta_R) = \iota_u(\mu_R(\Delta_R))$, which is zero when $\chi(M) = 0$.

Proposition 2.3.7. *The CDGAs* $\mathtt{Gra}_R^{\circlearrowleft}(U)$ *assemble to form a Hopf right* $\mathtt{Gra}_n^{\circlearrowleft}$*-comodule.*

Proof. The proof of this proposition is almost identical to the proof of Proposition 2.2.1. If we forget the extra differential (keeping only the internal differential of R), then $\mathtt{Gra}_R$ is the arity-wise tensor product $(\mathtt{Com}^\vee \circ R) \boxtimes \mathtt{Gra}_n$, which is automatically a Hopf $\mathtt{Gra}_n$-right comodule. Checking the compatibility with the differential involves almost exactly the same equations as Proposition 2.2.1, except that when $u, v \in W$ we have:

$$\circ_W^\vee(d(e_{uv})) = \iota_*(\mu_R(\Delta_R)) \otimes 1 = d(e_{**} \otimes 1 + 1 \otimes e_{uv}) = d(\circ_W^\vee(e_{uv})). \qquad \square$$

We now give a graphical interpretation of Definition 2.3.5, in the spirit of Section 2.3.1. We view $\mathtt{Gra}_R^{\circlearrowleft}(U)$ as spanned by graphs with U as set of vertices, and each vertex has a label which is an element of R. The $\mathtt{Gra}_n^{\circlearrowleft}$-comodule structure collapses subgraphs as before, and the label of the collapsed vertex is the product of all the labels in the subgraph.

Let $\Gamma \in \mathtt{Gra}_R^{\circlearrowleft}(U)$ be some graph. The differential of Γ, as defined in Definition 2.3.5, is the sum over the edges $e \in E_\Gamma$ of the graph $\Gamma - e$ with that edge removed and the labels of the endpoints multiplied by the factors of

$$\Delta_R = \sum_{(\Delta_R)} \Delta_R' \otimes \Delta_R'' \in R \otimes R.$$

In particular if e is a loop, then in the corresponding factor of $d\Gamma$ the vertex incident to e has its label multiplied by $\mu_R(\Delta_R)$. We will often write d_{split} for this differential, to contrast it with the differential that contracts edges which will occur in the complex $\mathrm{Tw}\,\mathtt{Gra}_R^{\circlearrowleft}$ defined later on. See Figure 2.3.2 for an example – gray vertices can be either internal or external.

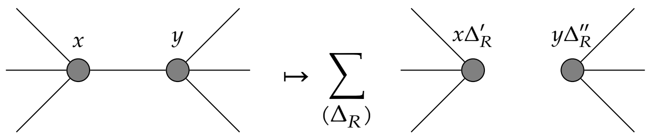


Figure 2.3.2: The splitting part of the differential (on one edge)

Finally the Hopf structure glues two graphs along their vertices, multiplying the labels in the process.

If $\chi(M) \neq 0$, we cannot directly map $\mathtt{Gra}_R^{\circlearrowleft}$ to $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_M)$, as the Euler class in $\Omega^*_{\mathrm{PA}}(M)$ would need to be the boundary of the image of the loop $e_{11} \in \mathtt{Gra}_R^{\circlearrowleft}(\underline{1})$. We thus define a sub-CDGA which will map to $\Omega^*_{\mathrm{PA}}(\mathsf{FM}_M)$ whether $\chi(M)$ vanishes or not.

Definition 2.3.8. For a given finite set U, let $\mathtt{Gra}_R(U)$ be the sub-CDGA of $\mathtt{Gra}_R^{\circlearrowleft}(U)$ spanned by graphs without loops.

Proposition 2.3.9. *The space $\mathtt{Gra}_R(U)$ is a sub-CDGA, and if $\chi(M) = 0$ the collection $\mathtt{Gra}_R$ assembles to form a Hopf right $\mathtt{Gra}_n$-comodule.*

Proof. Clearly, neither the splitting part of the differential nor the internal differential coming from R can create new loops, nor can the product of two graphs without loops contain a loop, thus $\mathtt{Gra}_R(U)$ is indeed a sub-CDGA of $\mathtt{Gra}_R^{\circlearrowleft}(U)$.

Let us now assume that $\chi(M) = 0$. The proof that $\mathtt{Gra}_R$ is a $\mathtt{Gra}_n$-comodule is now almost the same as the proof of Proposition 2.3.7, except that we need to use the fact that $\mu_R(\Delta_R) = 0$ to check that $d(\circ_W^\vee(e_{uv})) = \circ_W^\vee(d(e_{uv}))$ when $u, v \in W$. $\square$

2.3.3 The propagator

To define $\omega' : \mathtt{Gra}_R \to \Omega_{\mathrm{PA}}(\mathtt{FM}_M)$, we need a "propagator" $\varphi \in \Omega^{n-1}_{\mathrm{PA}}(\mathtt{FM}_M(2))$, for which a reference is [CM10, Section 4], who refine constructions of Axelrod–Singer [AS94].

For a given $u \in U$, we define the projection $p_u : \mathtt{FM}_M(U) \to M$ to be the map that forgets all the points of the configuration except the one labeled by u. The two projections $p_1, p_2 : \mathtt{FM}_M(\underline{2}) \to M$ are equal when restricted to the boundary, and form a sphere bundle $\partial\mathtt{FM}_M(\underline{2}) \to M$ ($\mathtt{FM}_M(2)$ is the blow-up of $M \times M$ along the diagonal). When M is framed, this bundle is trivial, and the operadic insertion map

$$M \times S^{n-1} \cong \mathtt{FM}_M(\underline{1}) \times \mathtt{FM}_n(\underline{2}) \xrightarrow{\circ_1} \partial\mathtt{FM}_M(2)$$

is an isomorphism of bundles.

Proposition 2.3.10 ([CW16, Proposition 7])**.** *There exists a form $\varphi \in \Omega^{n-1}_{\mathrm{PA}}(\mathtt{FM}_M(\underline{2}))$ such that $\varphi^{21} = (-1)^n\varphi$, $d\varphi = (p_1 \times p_2)^*(\sigma(\Delta_R))$ and such that the restriction of φ to $\partial\mathtt{FM}_M(2)$ is a global angular form, i.e. it is a volume form of S^{n-1} when restricted to each fiber. When M is framed one can moreover choose $\varphi|_{\partial\mathtt{FM}_M(2)} = 1 \times \mathrm{vol}_{S^{n-1}} \in \Omega^{n-1}_{PA}(M \times S^{n-1})$.*

One can see from the proofs of [CM10, Section 4] that $d\varphi$ can in fact be chosen to be any pullback of a form cohomologous to the diagonal class $\Delta_M \in \Omega^n_{\mathrm{PA}}(M \times M)$. We will make further adjustments to the propagator φ in Proposition 2.3.32.

Proposition 2.3.11. *There is a morphism of collections of CDGAs:*

$$\begin{aligned} \mathtt{Gra}_R &\xrightarrow{\omega'} \Omega_{\mathrm{PA}}(\mathtt{FM}_M) \\ \bigotimes_{u \in U} x_u \in R^{\otimes U} &\mapsto \bigwedge_{u \in U} p_u^*(\sigma(x_u)) \\ e_{uv} &\mapsto p_{uv}^*(\varphi), \end{aligned}$$

where p_{uv} was defined in Equation (2.1.6).

Moreover, if M is framed, then ω' defines a morphism of comodules:

$$(\mathtt{Gra}_R, \mathtt{Gra}_n) \xrightarrow{(\omega',\omega')} (\Omega^*_{\mathrm{PA}}(\mathtt{FM}_M), \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n))$$

where $\omega' : \mathtt{Gra}_n \to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n)$ was defined in Section 2.1.3.

Proof. The property $d\varphi = (p_1 \times p_2)^*(\Delta_R)$ shows that the map ω' preserves the differential.

Let us now assume that M is framed to prove that this is a morphism of right comodules. Cocomposition commutes with ω' on the generators coming from $A^{\otimes U}$, since the comodule structure of $\Omega^*_{\mathrm{PA}}(\mathtt{FM}_M)$ multiplies together forms that are pullbacks of forms on M:

$$\circ^\vee_W(p^*_u(x)) = \begin{cases} p^*_u(x) \otimes 1 & \text{if } u \notin W; \\ p^*_*(x) \otimes 1 & \text{if } u \in W. \end{cases}$$

We now check the compatibility of the cocomposition $\circ^\vee_W$ with ω' on the generator ω_{uv}, for some $W \subset U$.

- If one of u, v, or both, is not in W, then the equality

  $$\circ^\vee_W(\omega'(e_{uv})) = (\omega' \otimes \omega')(\circ^\vee_W(e_{uv}))$$

  is clear by the previous relation.

- Otherwise suppose $\{u, v\} \subset W$. We may assume that $U = W = \underline{2}$ (it suffices to pull back the result along p_{uv} to get the general case), so that we are considering the insertion of an infinitesimal configuration $M \times \mathtt{FM}_n(\underline{2}) \to \mathtt{FM}_M(\underline{2})$. This insertion factors through the boundary $\partial\mathtt{FM}_M(\underline{2})$. We have (recall Definition 2.3.10):

  $$\circ^\vee_{\underline{2}}(\varphi) = 1 \otimes \mathrm{vol}_{S^{n-1}} \in \Omega^*_{\mathrm{PA}}(M) \otimes \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n(\underline{2})) = \Omega^*_{\mathrm{PA}}(M) \otimes \Omega^*_{\mathrm{PA}}(S^{n-1}).$$

  Going back to the general case, we find:

  $$\circ^\vee_W(\omega'(e_{uv})) = \circ^\vee_W(p^*_{uv}(\varphi)) = 1 \otimes p^*_{uv}(\mathrm{vol}_{S^{n-1}}),$$

  which is indeed the image of $\circ^\vee_W(\omega_{uv}) = 1 \otimes \omega_{uv}$ by $\omega' \otimes \omega'$. □

2.3.4 Twisting: $\mathrm{Tw}\,\mathtt{Gra}_R$

The general framework of [Wil16, Appendix C] shows that to twist a right (co)module, one only needs to twist the (co)operad. As before, the condition on the arity zero component of the Hopf cooperad provides the Hopf structure on the twisted comodule.

Definition 2.3.12. The **twisted labeled graph comodule** $\mathrm{Tw}\,\mathtt{Gra}^{\circlearrowleft}_R$ is a Hopf right comodule over $(\mathrm{Tw}\,\mathtt{Gra}^{\circlearrowleft}_n)$, obtained from $\mathtt{Gra}^{\circlearrowleft}_R$ by twisting with respect to the Maurer–Cartan element $\mu \in (\mathtt{Gra}^{\circlearrowleft}_n)^\vee(\underline{2})$ of Section 2.1.3.

We now explicitly describe this comodule in terms of graphs.

The dg-module $\mathrm{Tw}\,\mathtt{Gra}_R^{\circlearrowleft}(U)$ is spanned by graphs with two kinds of vertices, external vertices corresponding to elements of U, and indistinguishable internal vertices (usually drawn in black). The degree of an edge is still $n-1$, while the degree of an internal vertex is $-n$. All the vertices are labeled by elements of R, and their degree is added to the degree of the graph.

The Hopf structure glues two graphs along their external vertices, multiplying labels in the process. The differential is a sum of two terms $d_{\mathrm{split}} + d_{\mathrm{contr}}$ (in addition to the internal differential coming from R). The first part comes from $\mathtt{Gra}_R^{\circlearrowleft}$ and splits edges, multiplying by Δ_R the labels of the endpoints. The second part is similar to the differential of $\mathrm{Tw}\,\mathtt{Gra}_n^{\circlearrowleft}$: it contracts edges connecting an internal vertex to another vertex of either kind, multiplying the labels of the endpoints (see Figure 2.1.3).

Remark 2.3.13. Unlike $\mathrm{Tw}\,\mathtt{Gra}_n$, dead ends *are* contractible in $\mathrm{Tw}\,\mathtt{Gra}_R^{\circlearrowleft}$. This is because the Maurer–Cartan element in $(\mathtt{Gra}_n^{\circlearrowleft})^\vee(\underline{2})$ can only "act" while coming from the right side on $\mathtt{Gra}_R^{\circlearrowleft}$ in the definition of the differential, and so there is nothing to cancel out the contraction of a dead end.

Finally, the comodule structure is similar to the cooperad structure of $\mathrm{Tw}\,\mathtt{Gra}_n^{\circlearrowleft}$: for $\Gamma \in \mathtt{Gra}_R^{\circlearrowleft}(U \sqcup I) \subset \mathrm{Tw}\,\mathtt{Gra}_R^{\circlearrowleft}(U)$, the cocomposition $\circ_W^\vee(\Gamma)$ is the sum over tensors of the type $\pm\Gamma_{U/W} \otimes \Gamma_W$, where $\Gamma_{U/W} \in \mathtt{Gra}_R^{\circlearrowleft}(U/W \sqcup J)$, $\Gamma_W \in \mathtt{Gra}_n(W \sqcup J')$, $J \sqcup J' = I$, and there exists a way of inserting Γ_W in the vertex $*$ of $\Gamma_{U/W}$ and reconnecting edges to get back Γ. See Figure 2.3.3 for an example.

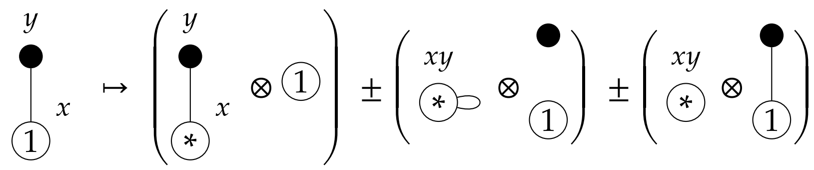


Figure 2.3.3: Example of cocomposition $\mathrm{Tw}\,\mathtt{Gra}_R^{\circlearrowleft}(\underline{1}) \to \mathrm{Tw}\,\mathtt{Gra}_R^{\circlearrowleft}(\underline{1}) \otimes \mathrm{Tw}\,\mathtt{Gra}_n^{\circlearrowleft}(\underline{1})$

Lemma 2.3.14. *The subspace* $\mathrm{Tw}\,\mathtt{Gra}_R(U) \subset \mathrm{Tw}\,\mathtt{Gra}_R^{\circlearrowleft}(U)$ *spanned by graphs with no loops is a sub-CDGA.*

Proof. It is clear that this defines a subalgebra. We need to check that it is preserved by the differential, i.e. that the differential cannot create new loops if there are none in a graph. This is clear for the internal differential coming from R and for the splitting part of the differential. The contracting part of the differential could create a loop from a double edge; however for even n multiple edges are zero for degree reasons, and for odd n loops are zero because of the antisymmetry relation (see Remark 2.3.6). □

Note that despite the notation, $\mathrm{Tw}\,\mathtt{Gra}_R$ is a priori not defined as the twisting of the $\mathtt{Gra}_n$-comodule $\mathtt{Gra}_R$: when $\chi(M) \neq 0$, the collection $\mathtt{Gra}_R$ is not even a $\mathtt{Gra}_n$-comodule. However, the following proposition is clear and shows that we can get away with this abuse of notation:

Proposition 2.3.15. *If $\chi(M) = 0$, then $\mathrm{Tw}\,\mathtt{Gra}_R$ assembles to a right Hopf comodule over $(\mathrm{Tw}\,\mathtt{Gra}_n)$, isomorphic to the twisting of the right Hopf $\mathtt{Gra}_n$-comodule $\mathtt{Gra}_R$ of Definition 2.3.8.* □

Remark 2.3.16. We could have defined the algebra $\mathrm{Tw}\,\mathtt{Gra}_R$ explicitly in terms of labeled graphs, and then defined the differential d using an ad-hoc formula. The difficult part would have then been to check that $d^2 = 0$ (involving difficult signs), which is a consequence of the general operadic twisting framework.

Proposition 2.3.17. *There is a morphism of collections of CDGAs $\omega : \mathrm{Tw}\,\mathtt{Gra}_R \to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M)$ extending ω', given on a graph $\Gamma \in \mathtt{Gra}_R(U \sqcup I) \subset \mathrm{Tw}\,\mathtt{Gra}_R(U)$ by:*

$$\omega(\Gamma) := (p_U)_*(\omega'(\Gamma)) = \int_{\mathtt{FM}_M(U \sqcup I) \to \mathtt{FM}_M(U)} \omega'(\Gamma).$$

Moreover, if M is framed, then this defines a morphism of Hopf right comodules:

$$(\omega, \omega) : (\mathrm{Tw}\,\mathtt{Gra}_R, \mathrm{Tw}\,\mathtt{Gra}_n) \to (\Omega^*_{\mathrm{PA}}(\mathtt{FM}_M), \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n)).$$

Remark 2.3.18. It is not a priori possible to integrate any arbitrary form along the fiber of the projection p_U, see [Har+11, Section 9.4]. However, following [CW16, Appendix C], we can assume that the morphism $\sigma : R \to \Omega^*_{\mathrm{PA}}(M)$ factors through the quasi-isomorphism sub-CDGA of "trivial forms" and that the propagator is a trivial form. This makes $\omega(\Gamma)$ well-defined.

Proof. The proof of the compatibility with the Hopf structure and, in the framed case, the comodule structure, is formally similar to the proof of the same facts about $\omega : \mathrm{Tw}\,\mathtt{Gra}_n \to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_n)$. We refer to [LV14, Sections 9.2, 9.5]. The proof is exactly the same proof, but writing $\mathtt{FM}_M$ or $\mathtt{FM}_n$ instead of $C[-]$ and φ instead of $\mathrm{vol}_{S^{n-1}}$ in every relevant sentence, and recalling that when M is framed, we choose φ such that $\circ^\vee_{\underline{2}}(\varphi) = 1 \otimes \mathrm{vol}_{S^{n-1}}$.

The proof that ω is a chain map is different albeit similar. The rest of the section is dedicated to that proof. □

We recall Stokes' formula for integrals along fibers of semi-algebraic bundles. If $\pi : E \to B$ is a semi-algebraic bundle, the fiberwise boundary $\pi^\partial : E^\partial \to B$ is the bundle with total space

$$E^\partial = \bigcup_{b \in B} \partial \pi^{-1}(b).$$

Remark 2.3.19. The space E^∂ is neither ∂E nor $\bigcup_{b \in B} \pi^{-1}(b) \cap \partial E$ in general. Consider for example the projection on the first coordinate $[0,1]^{\times 2} \to [0,1]$.

Stokes' formula is given in the semi-algebraic context by [Har+11, Proposition 8.12]:

$$d\left(\int_{\pi : E \to B} \alpha\right) = \int_{\pi : E \to B} d\alpha \pm \int_{\pi^\partial : E^\partial \to B} \alpha|_{E^\partial}.$$

If we apply this formula to compute $d\omega(\Gamma)$, we find that the first part is given by:

$$\int_{p_U} d\omega'(\Gamma) = \int_{p_U} \omega'(d_R\Gamma + d_{\mathrm{split}}\Gamma) = \omega(d_R\Gamma + d_{\mathrm{split}}\Gamma), \tag{2.3.20}$$

since ω' was a chain map. It thus remain to check that the second summand satisfies:

$$\int_{p_U^\partial : \mathsf{FM}_M^\partial(U \sqcup I) \to \mathsf{FM}_M(U)} \omega'(\Gamma) = \int_{p_U} \omega'(d_{\mathrm{contr}}\Gamma) = \omega(d_{\mathrm{contr}}\Gamma).$$

The fiberwise boundary of the projection $p_U : \mathsf{FM}_n(U \sqcup I) \to \mathsf{FM}_n(U)$ is rather complex [LV14, Section 5.7], essentially due to the quotient by the affine group in the definition of FM_n which lowers dimensions. We will not repeat its explicit decomposition into cells as we do not need it here.

The fiberwise boundary of $p_U : \mathsf{FM}_M(U \sqcup I) \to \mathsf{FM}_M(U)$ is simpler. Let $V = U \sqcup I$. The interior of $\mathsf{FM}_M(U)$ is the space $\mathrm{Conf}_U(M)$, and thus $\mathsf{FM}_M^\partial(V)$ is the closure of $(\partial \mathsf{FM}_M(V)) \cap \pi^{-1}(\mathrm{Conf}_U(M))$. Let

$$\mathcal{BF}(V, U) = \{W \subset V \mid \#W \geq 2 \text{ and } \#W \cap U \leq 1\}.$$

Lemma 2.3.21. *The subspace $\mathsf{FM}_M^\partial(V) \subset \mathsf{FM}_M(V)$ is equal to:*

$$\bigcup_{W \in \mathcal{BF}(V,U)} \mathrm{im}(\circ_W : \mathsf{FM}_M(V/W) \times \mathsf{FM}_n(W) \to \mathsf{FM}_M(V)).$$

In the description of $\mathsf{FM}_n^\partial(V)$, there was an additional part which corresponds to $U \subset W$. But unlike FM_n, for FM_M the image of $p_U(- \circ_W -)$ is always included in the boundary of $\mathsf{FM}_M(U)$ when $U \subset W$. We follow a pattern similar to the one used in the proof of [LV14, Proposition 5.7.1].

Proof. Let cls denote the closure operator. Since $\mathrm{Conf}_U(M)$ is the interior of $\mathsf{FM}_M(U)$ and $p : \mathsf{FM}_M(V) \to \mathsf{FM}_M(U)$ is a bundle, it follows that:

$$\begin{aligned} \mathsf{FM}_M^\partial(V) &= \mathrm{cls}(\mathsf{FM}_M^\partial(V) \cap p^{-1}(\mathrm{Conf}_U(M))) \\ &= \mathrm{cls}(\partial \mathsf{FM}_M(V) \cap p^{-1}(\mathrm{Conf}_U(M))). \end{aligned}$$

The boundary $\partial\mathsf{FM}_M(V)$ is the union of the $\mathrm{im}(\circ_W)$ for $\#W \geq 2$ (note that the case $W = V$ is included, unlike for FM_n). If $\#W \cap U \geq 2 \iff W \notin \mathcal{BF}(V, U)$, then $\mathrm{im}(p_U(- \circ_W -)) \subset \partial\mathsf{FM}_M(U)$, because if a configuration belongs to this image then at least two points of U are infinitesimally close. Therefore:

$$\begin{aligned}
\mathsf{FM}_M^\partial(V) &= \mathrm{cls}(\partial\mathsf{FM}_M(V) \cap p^{-1}(\mathrm{Conf}_U(M))) \\
&= \mathrm{cls}\Big(\bigcup_{\#W\geq 2} \mathrm{im}(\circ_W) \cap \pi^{-1}(\mathrm{Conf}_U(M))\Big) \\
&= \mathrm{cls}\Big(\bigcup_{\#W\in\mathcal{BF}(V,U)} \mathrm{im}(\circ_W) \cap \pi^{-1}(\mathrm{Conf}_U(M))\Big) \\
&= \bigcup_{W\in\mathcal{BF}(V,U)} \mathrm{cls}(\mathrm{im}(\circ_W) \cap \pi^{-1}(\mathrm{Conf}_U(M))) \\
&= \bigcup_{W\in\mathcal{BF}(V,U)} \mathrm{im}(\circ_W).
\end{aligned}$$

□

Lemma 2.3.22. *For a given graph $\Gamma \in \mathrm{Tw}\,\mathsf{Gra}_R(U)$, the integral over the fiberwise boundary is given by:*

$$\int_{p_U^\partial} \omega'(\Gamma)|_{\mathsf{FM}_M^\partial(V)} = \omega(d_{\mathrm{contr}}\Gamma).$$

Proof. The maps $\circ_W : \mathsf{FM}_M(V/W) \times \mathsf{FM}_n(W) \to \mathsf{FM}_M(V)$ are smooth injective map and their domains are compact, thus they are homeomorphisms onto their images. Recall $\#W \geq 2$ for $W \in \mathcal{BF}(V, U)$, hence $\dim \mathsf{FM}_n(W) = n\#W - n - 1$. The dimension of the image of $\circ_W$ is then:

$$\begin{aligned}
\dim \mathrm{im}(\circ_W) &= \dim \mathsf{FM}_M(V/W) + \dim \mathsf{FM}_n(W) \\
&= n\#(V/W) + (n\#W - n - 1) \\
&= n\#V - 1,
\end{aligned} \tag{2.3.23}$$

i.e. the image is of codimension 1 in $\mathsf{FM}_M(V)$. It is also easy to check that if $W \neq W'$, then $\mathrm{im}(\circ_W) \cap \mathrm{im}(\circ_{W'})$ is of codimension strictly bigger than 1.

We now fix $W \in \mathcal{BF}(V, U)$. Since $\#W \cap U \leq 1$, the composition $U \subset V \to V/W$ is injective and identifies U with a subset of V/W. There is then a forgetful map $p'_U : \mathsf{FM}_M(V/W) \to \mathsf{FM}_M(U)$. We then have a commutative diagram:

$$\begin{tikzcd}
\mathsf{FM}_M(V/W) \times \mathsf{FM}_n(W) \arrow[r, "p_1"] \arrow[d, "\circ_W"] & \mathsf{FM}_M(V/W) \arrow[d, "p'_U"] \\
\mathsf{FM}_M(V) \arrow[r, "p_U"] & \mathsf{FM}_M(U)
\end{tikzcd} \tag{2.3.24}$$

It follows that $p_U(- \circ_W -) = p'_U \circ p_1$ is the composite of two semi-algebraic bundles, hence it is a semi-algebraic bundle itself [Har+11, Proposition 8.5].

Combined with the fact about codimensions above, we can therefore apply the summation formula [Har+11, Proposition 8.11]:

$$\int_{p_U^\partial} \omega'(\Gamma) = \sum_{W \in \mathcal{BF}(V,U)} \int_{p_U(-\circ_W -)} \omega'(\Gamma)|_{\mathsf{FM}_M(V/W)\times \mathsf{FM}_n(W)} \tag{2.3.25}$$

Now we can directly adapt the proof of Lambrechts and Volić. For a fixed W, by [Har+11, Proposition 8.13], the corresponding summand is equal to $\pm\omega(\Gamma_{V/W}) \cdot \int_{\mathsf{FM}_n(W)} \omega'(\Gamma_W)$, where

- $\Gamma_{V/W} \in \mathrm{Tw}\,\mathsf{Gra}_R(U)$ is the graph with W collapsed to a vertex and $U \hookrightarrow V/W$ is identified with its image;
- $\Gamma_W \in \mathrm{Tw}\,\mathsf{Gra}_n(W)$ is the full subgraph of Γ with vertices W and the labels removed.

The vanishing lemmas in the proof of Lambrechts and Volić then imply that the integral $\int_{\mathsf{FM}_n(W)} \omega'(\Gamma_W)$ is zero unless Γ_W is the graph with exactly two vertices and one edge, in which case the integral is equal to 1. In this case, $\Gamma_{V/W}$ is the graph Γ with one edge connecting an internal vertex to some other vertex collapsed. The sum runs over all such edges, and dealing with signs carefully we see that Equation (2.3.25) is precisely equal to $\omega(d_{\mathrm{contr}}\Gamma)$. □

End of the proof of Proposition 2.3.17. By Equation (2.3.20) and Lemma 2.3.22, we can apply Stokes' formula to $d\omega(\Gamma)$ to show that it is equal to $\omega(d\Gamma) = \omega(d_R\Gamma + d_{\mathrm{split}}\Gamma) + \omega(d_{\mathrm{split}}\Gamma)$. □

2.3.5 Reduction: Graphs_R

The last step in the construction of Graphs_R is the reduction of $\mathrm{Tw}\,\mathsf{Gra}_R$ so that it has the right cohomology. We borrow the terminology of Campos–Willwacher [CW16] for the next two definitions.

Definition 2.3.26. Let the **full graph complex** be:

$$\mathrm{fGC}_R = \mathrm{Tw}\,\mathsf{Gra}_R(\emptyset)[-n].$$

It consists of graphs with only internal vertices, and the product is disjoint union of graphs. The degree of a graph γ is $n + (n-1)\#E_\gamma - n\#V_\gamma$, where E_γ is the set of edges and V_γ the set of vertices.

Remark 2.3.27. The degree shift is there to be consistent with the definition of the standard graph complex GC_n.

As an algebra, $\mathrm{fGC}_R[n]$ is free and generated by connected graphs. In general we will call "internal components" the connected components of a graph that only contain internal vertices. The full graph complex naturally acts on $\mathrm{Tw}\,\mathtt{Gra}_R(U)$ by adding extra internal components.

Definition 2.3.28. The **partition function** $Z_\varphi : \mathrm{fGC}_R[n] \to \mathbb{R}$ is the restriction:

$$Z_\varphi = \omega|_\emptyset : \mathrm{Tw}\,\mathtt{Gra}_R(\emptyset) = \mathrm{fGC}_R[n] \to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M(\emptyset)) = \Omega^*_{\mathrm{PA}}(\mathrm{pt}) = \mathbb{R}.$$

By the double-pushforward formula and Fubini's theorem, Z_φ is an algebra morphism and

$$\forall \gamma \in \mathrm{fGC}_R[n], \forall \Gamma \in \mathrm{Tw}\,\mathtt{Gra}_R(U),\ \omega(\gamma \cdot \Gamma) = Z_\varphi(\gamma) \cdot \omega(\Gamma). \tag{2.3.29}$$

Definition 2.3.30. Let $\mathbb{R}_\varphi$ be the $\mathrm{fGC}_R[n]$-module of dimension 1 induced by $Z_\varphi : \mathrm{fGC}_R[n] \to \mathbb{R}$. The **reduced graph comodule** $\mathtt{Graphs}^\varphi_R$ is the tensor product:

$$\mathtt{Graphs}^\varphi_R(U) = \mathbb{R}_\varphi \otimes_{\mathrm{fGC}_R[n]} \mathrm{Tw}\,\mathtt{Gra}_R(U).$$

In other words, a graph of the type $\Gamma \sqcup \gamma$ containing an internal component $\gamma \in \mathrm{fGC}_R[n]$ is identified with $Z_\varphi(\gamma) \cdot \Gamma$. It is spanned by representative classes of graphs with no internal connected component; we call such graphs **reduced**. The notation is meant to evoke the fact that $\mathtt{Graphs}^\varphi_R$ depends on the choice of the propagator φ, unlike the collection $\mathtt{Graphs}^\varepsilon_R$ that will appear in Section 2.4.1.

Proposition 2.3.31. *The map $\omega : \mathrm{Tw}\,\mathtt{Gra}_R(U) \to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M(U))$ defined in Proposition 2.3.17 factors through the quotient defining $\mathtt{Graphs}^\varphi_R$.*

If $\chi(M) = 0$, the symmetric collection $\mathtt{Graphs}^\varphi_R$ forms a Hopf right comodule over $\mathtt{Graphs}_n$. If moreover M is framed, the map ω defines a Hopf right comodule morphism.

Proof. Equation (2.3.29) immediately implies that ω factors through the quotient.

The vanishing lemmas shows that if $\Gamma \in \mathrm{Tw}\,\mathtt{Gra}_n(U)$ has internal components, then $\omega(\Gamma)$ vanishes [LV14, Proposition 9.3.1], so it is straightforward to check that if $\chi(M) = 0$, then $\mathtt{Graphs}^\varphi_R$ becomes a Hopf right comodule over $\mathtt{Graphs}_n$. It is also clear that for M framed, the quotient map ω remains a Hopf right comodule morphism. □

Proposition 2.3.32 ([CM10, Lemma 3])**.** *The propagator φ can be chosen such that the additional property (P4) holds:*

$$\int_{p_1 : \mathtt{FM}_M(\underline{2}) \to \mathtt{FM}_M(\underline{1}) = M} p_2^*(\sigma(x)) \wedge \varphi = 0,\ \ \forall x \in R; \tag{P4}$$

Remark 2.3.33. The additional property (P5) of the paper mentioned above would be helpful in order to get a direct morphism $\mathtt{Graphs}_R^\varphi \to \mathtt{G}_A$, because then the partition function would vanish on all connected graphs with at least two vertices. However we run into difficulties when trying to adapt the proof in the setting of PA forms, mainly due to the lack of an operator d_M acting on $\Omega^*_{\mathrm{PA}}(M \times N)$ differentiating "only in the first slot".

From now on and until the end of the paper, we assume that φ satisfies (P4).

Corollary 2.3.34. *The morphism ω vanishes on graphs containing univalent internal vertices (i.e. dead ends).*

Proof. Let $\Gamma \in \mathtt{Gra}_R(U \sqcup I) \subset \mathrm{Tw}\,\mathtt{Gra}_R(U)$ be a graph with a univalent internal vertex $u \in I$, labeled by x, and let v be the only vertex connected to u. Let $\tilde{\Gamma}$ be the full subgraph of Γ on the set of vertices $U \sqcup I - \{u\}$. Then using [Har+11, Propositions 8.10 and 8.15] (in a way similar to the end of the proof of [LV14, Lemma 9.3.9]), we find:

$$\begin{aligned}\omega(\Gamma) &= \int_{\mathtt{FM}_M(U\sqcup I)\to\mathtt{FM}_M(U)} \omega'(\Gamma) \\ &= \int_{\mathtt{FM}_M(U\sqcup I)\to\mathtt{FM}_M(U)} \omega'(\tilde{\Gamma}) p^*_{uv}(\varphi) p^*_u(\sigma(x)) \\ &= \int_{\mathtt{FM}_M(U\sqcup I-\{u\})\to\mathtt{FM}_M(U)} \omega'(\tilde{\Gamma}) \wedge p^*_v\left(\int_{\mathtt{FM}_M(\{u,v\})\to\mathtt{FM}_M(\{v\})} p^*_{uv}(\varphi) p^*_u(\sigma(x))\right),\end{aligned}$$

which vanishes by (P4). □

Almost everything we have done so far works for non-simply connected manifolds. We now prove a proposition which sets simply connected manifolds apart.

Proposition 2.3.35. *The partition function Z_φ vanishes on any connected graph with no bivalent vertices labeled by 1_R and containing at least two vertices.*

Remark 2.3.36. If $\gamma \in \mathrm{fGC}_R$ has only one vertex, labeled by x, then $Z_\varphi(\gamma) = \int_M \sigma(x)$ which can be nonzero.

Proof. Let $\gamma \in \mathrm{fGC}_R[n]$ be a connected graph with at least two vertices and no bivalent vertices labeled by 1_R. By Corollary 2.3.34, we can assume that all the vertices of γ are at least bivalent. By hypothesis, if a vertex is bivalent then it is labeled by an element of $R^{>0} = R^{\geq 2}$.

Let $k = i + j$ be the number of vertices of γ, with i vertices that are at least trivalent and j vertices that are bivalent and labeled by $R^{\geq 2}$. It follows that γ has at least $\frac{1}{2}(3i + 2j)$ edges, all of degree $n - 1$. Since bivalent vertices are labeled by $R^{\geq 2}$, their labels contribute at least $2j$ to the degree of γ. The (internal) vertices contribute $-kn$ to the degree, and the other labels have a nonnegative contribution. Thus:

$$\begin{aligned}\deg\gamma &\geq \left(\frac{3}{2}i+j\right)(n-1)+2j-kn\\ &= \left(\frac{3}{2}k-\frac{3}{2}j+j\right)(n-1)+2j-kn\\ &= \frac{1}{2}(k(n-3)-j(n-5)).\end{aligned}$$

This last number is always positive for $0 \leq j \leq k$: it is an affine function of j, and it is positive when $j = 0$ and $j = k$ (recall that $n \geq 4$). The degree of $\gamma \in \mathrm{fGC}_R[n]$ must be zero for the integral defining $Z_\varphi(\gamma)$ to be the integral of a top form of $\mathtt{FM}_M(\underline{k})$ and hence possibly nonzero. But by the above computation, $\deg\gamma > 0 \implies Z_\varphi(\gamma) = 0$. □

Remark 2.3.37. When $n = 3$, the manifold M is the 3-sphere S^3 thanks to Perelman's proof [Per02; Per03] of the Poincaré conjecture. The partition function Z_φ is conjectured to be trivial on S^3 for a proper choice of framing, thus bypassing the need for the above degree counting argument. See also Proposition 2.4.37.

2.4 From the model to forms via graphs

In this section we connect $\mathtt{G}_A$ to $\Omega^*_{\mathrm{PA}}(\mathtt{FM}_M)$ and we prove that the connecting morphisms are quasi-isomorphisms.

2.4.1 Construction of the morphism to $\mathtt{G}_A$

Proposition 2.4.1. *For each finite set U, there is a CDGA morphism $\rho'_* : \mathtt{Gra}_R(U) \to \mathtt{G}_A(U)$ given by ρ on the $R^{\otimes U}$ factor and sending the generators e_{uv} to ω_{uv} on the $\mathtt{Gra}_n$ factor. When $\chi(M) = 0$, this defines a Hopf right comodule morphism $(\mathtt{Gra}_R, \mathtt{Gra}_n) \to (\mathtt{G}_A, \mathtt{e}_n^\vee)$.* □

If we could find a propagator for which property (P5) held (see Remark 2.3.33), then we could just send all graphs containing internal vertices to zero and obtain an extension $\mathtt{Graphs}_R^\varphi \to \mathtt{G}_A$. Since we cannot assume that (P5) holds, the definition of the extension is more complex. However we still have Proposition 2.3.35, and homotopically speaking, graphs with bivalent vertices are irrelevant.

Definition 2.4.2. Let fGC_R^0 be the quotient of fGC_R defined by identifying a disconnected vertex labeled by x with the number $\varepsilon(\rho(x)) = \int_M \sigma(x)$.

It's clear that Z_φ factors through a map $\mathsf{fGC}_R^0[n] \to \mathbb{R}$, for which we will keep the same notation Z_φ.

Lemma 2.4.3. *The subspace $I \subset \mathsf{fGC}_R^0$ spanned by graphs with at least one univalent vertex, or at least one bivalent vertex labeled by 1_R, or at least one label in $\ker(\rho : R \to A)$, is a (shifted) CDGA ideal.*

Proof. It is clear that I is an algebra ideal. Let us prove that it is a differential ideal. If one of the labels of Γ is in $\ker\rho$, then so do all the summands of $d\Gamma$, because $\ker\rho$ is a CDGA ideal of R.

If Γ contains a bivalent vertex u labeled by 1_R, then so does $d_R\Gamma$. In $d_{\mathrm{split}}\Gamma$, splitting one of the two edges connected to u produces a univalent vertex; and in $d_{\mathrm{contr}}\Gamma$, the contraction of the two edges connected to u cancel each other.

Finally let us prove that if Γ has a univalent vertex u, then $d\Gamma$ lies in I. It's clear that $d_R\Gamma \in I$. Contracting or splitting the only edge connected to the univalent vertex could remove the univalent vertex. Let us prove that these two summands cancel each other up to $\ker\rho$.

Let x be the label of u, and let y be the label of the only vertex incident to u. Contracting the edge yields a new vertex labeled by xy. Due to the definition of fGC_R^0, splitting the edge yields a new vertex labeled by

$$\begin{aligned} \alpha &= \sum_{(\Delta_R)} \varepsilon(\rho(y\Delta_R''))x\Delta_R' \\ \implies \rho(\alpha) &= \rho(x) \cdot \sum_{(\Delta_A)} \pm\varepsilon_A(\rho(y)\Delta_A'')\Delta_A'. \end{aligned}$$

It is a standard property of the diagonal class Δ_A that $\sum_{(\Delta_A)} \pm\varepsilon_A(a\Delta_A'')\Delta_A' = a$ for all $a \in A$, directly from Equation (2.1.18). Applied to $a = \rho(y)$, it follows from the previous equation $\rho(\alpha) = \pm\rho(xy)$; examining the signs, this summand cancels from the summand that comes from contracting the edge. □

Definition 2.4.4. The algebra fGC_R' is the quotient of fGC_R^0 by the ideal I.

Note that $\mathsf{fGC}_R'[n]$ is also free as an algebra, with generators given by connected graphs with no univalent vertices nor bivalent vertices labeled by 1_R, and where the labels in $R/\ker(\rho) = A$.

Definition 2.4.5. Let $\mathsf{fLoop}_R \subset \mathsf{fGC}_R^0$ be the sub-CDGA generated by graphs with univalent vertices and by circular graphs (i.e. graphs of the type $e_{12}e_{23} \dots e_{(k-1)k}e_{k1}$).

Lemma 2.4.6. *The morphism* $\mathrm{Z}_\varphi : \mathrm{fGC}^0_R[n] \to \mathbb{R}$ *vanishes on* $\mathrm{fLoop}_R[n]$.

Proof. The map Z_φ vanishes on graphs with univalent vertices by Corollary 2.3.34. The degree of a circular graph with k vertices is $-k < 0$, but Z_φ can only be nonzero on graphs of degree zero. □

Proposition 2.4.7. *The sequence* $\mathrm{fLoop}_R \to \mathrm{fGC}^0_R \to \mathrm{fGC}'_R$ *is a homotopy cofiber sequence.*

Proof. The underlying algebra of fGC^0_R is a quasi-free extension of fLoop_R by the algebra generated by graphs that are not circular and that do not contain any univalent vertices. The homotopy cofiber of the inclusion $\mathrm{fLoop}_R \to \mathrm{fGC}^0_R$ is this algebra fGC''_R, together with a differential induced by the quotient $\mathrm{fGC}^0_R/(\mathrm{fLoop}_R)$. We aim to prove that the induced morphism $\mathrm{fGC}''_R \to \mathrm{fGC}'_R$ is a quasi-isomorphism.

Let us define an increasing filtration on both CDGAs by letting $F_s\mathrm{fGC}'_R$ (resp. $F_s\mathrm{fGC}''_R$) be the submodule spanned by graphs Γ such that $\#\text{edges} - \#\text{vertices} \leq s$ (see also the proof of Proposition 2.4.17 where a similar technique is reused).

The splitting part of the differential strictly decreases the filtration, so only d_R and d_{contr} remain on the first page of the associated spectral sequences.

One can then filter by the number of edges. On the first page of the spectral sequence associated to this new filtration, there is only the internal differential d_R. Thus on the second page, the vertices are labeled by $H^*(R) = H^*(M)$. The contracting part of the differential decreases the new filtration by exactly one, and so on the second page we see all of d_{contr}.

We can now adapt the proof of [Wil14, Proposition 3.4] to show that on the part of the complex with bivalent vertices, only the circular graphs contribute to the cohomology (we work dually so we consider a quotient instead of an ideal, but the idea is the same). To adapt the proof, one must see the labels of positive degree as formally adding one to the valence of the vertex, thus "breaking" a line of bivalent vertices. These label break the symmetry (recall the coinvariants in the definition of the twisting) that allow cohomology classes to be produced. □

Corollary 2.4.8. *The morphism* $\mathrm{fGC}_R[n] \to \mathbb{R}$ *factors through* $\mathrm{fGC}'_R[n]$ *up to homotopy.* □

Let $\pi : \mathrm{fGC}_R \to \mathrm{fGC}'_R$ be the quotient map. Let also $\Omega^*(\Delta^1) = S(t, dt)$ be the algebra of polynomials forms on Δ^1, which is a path object for $\mathbb{R}$ in the model category of CDGAs.

The CDGAs $\mathrm{fGC}_R[n]$ and $\mathrm{fGC}'_R[n]$ are both quasi-free with a good filtration: the generators are graphs (with some conditions for $\mathrm{fGC}'_R[n]$), and the filtration is given by the number of edges. Therefore they are cofibrant as CDGAs. Thus there exists some morphism $Z'_\varphi : \mathrm{fGC}'_R[n] \to \mathbb{R}$ and some homotopy $h : \mathrm{fGC}_R[n] \to \Omega^*(\Delta^1)$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathrm{fGC}_R[n] \arrow[dl, "Z_\varphi"'] \arrow[d, "h"] \arrow[dr, "Z'_\varphi \pi"] & \\ \mathbb{R} & \Omega^*(\Delta^1) \arrow[l, "d_1", "\sim"'] \arrow[r, "d_0"', "\sim"] & \mathbb{R} \end{tikzcd}$$

Definition 2.4.9. Let $\Omega^*(\Delta^1)_h$ be the $\mathrm{fGC}_R[n]$-module induced by h, and let

$$\mathsf{Graphs}'_R(U) = \Omega^*(\Delta^1)_h \otimes_{\mathrm{fGC}_R[n]} \mathrm{Tw}\,\mathsf{Gra}_R(U).$$

Definition 2.4.10. Let $Z_\varepsilon : \mathrm{fGC}_R[n] \to \mathbb{R}$ be the algebra morphism that sends a graph Γ with a single vertex labeled by $x \in R$ to $\int_M \sigma(x) = \varepsilon(\rho(x))$, and that sends all the other connected graphs to zero. Let $\mathbb{R}_\varepsilon$ be the one-dimensional $\mathrm{fGC}_R[n]$-module induced by Z_ε, and let

$$\mathsf{Graphs}^\varepsilon_R(U) = \mathbb{R}_\varepsilon \otimes_{\mathrm{fGC}_R[n]} \mathrm{Tw}\,\mathsf{Gra}_R(U).$$

Explicitly, in $\mathsf{Graphs}^\varepsilon_R$, all internal components with at least two vertices are identified with zero, whereas an internal component with a single vertex labeled by $x \in R$ is identified with the number $\int_M \sigma(x) = \varepsilon(\rho(x))$.

Lemma 2.4.11. *The morphism $Z'_\varphi \pi$ is equal to Z_ε.*

Proof. This is a rephrasing of Proposition 2.3.35. Using the same degree counting argument, all the connected graphs with more than one vertex in $\mathrm{fGC}'_R[n]$ are of positive degree. Since $\mathbb{R}$ is concentrated in degree zero, $Z'_\varphi \pi$ must vanish on these graphs, just like Z_ε.

Besides the morphism $\pi : \mathrm{fGC}_R \to \mathrm{fGC}'_R = \mathrm{fGC}^0_R / I$ factors through fGC^0_R, where graphs γ with a single vertex are already identified with the numbers $Z_\varphi(\gamma) = Z_\varepsilon(\gamma)$. □

Proposition 2.4.12. *For each finite set U, we have a zigzag of quasi-isomorphisms of CDGAs:*

$$\mathsf{Graphs}^\varepsilon_R(U) \xleftarrow{\sim} \mathsf{Graphs}'_R(U) \xrightarrow{\sim} \mathsf{Graphs}^\varphi_R(U).$$

If $\chi(M) = 0$, then Graphs'_R and $\mathsf{Graphs}^\varepsilon_R$ are right Hopf Graphs_n-comodules, and the zigzag defines a zigzag of Hopf right comodule morphisms.

Proof. We have two commutative diagrams:

$$\begin{tikzcd} \mathsf{Graphs}^\varepsilon_R(U) & & \mathsf{Graphs}'_R(U) \arrow[ll] \\ \mathrm{Tw}\,\mathsf{Gra}_R(U) \otimes_{\mathrm{fGC}_R[n]} \mathbb{R}_\varepsilon \arrow[u, "="] & & \mathrm{Tw}\,\mathsf{Gra}_R(U) \otimes_{\mathrm{fGC}_R[n]} \Omega^*(\Delta^1)_h \arrow[ll, "1 \otimes d_1"'] \arrow[u, "="] \end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Graphs}'_R(U) \arrow[rr] & & \mathsf{Graphs}^{\varphi}_R(U) \\
\mathrm{Tw}\,\mathsf{Gra}_R(U) \otimes_{\mathsf{fGC}_R[n]} \Omega^*(\Delta^1)_h \arrow[u, "="] \arrow[rr, "1\otimes d_0"] & & \mathrm{Tw}\,\mathsf{Gra}_R(U) \otimes_{\mathsf{fGC}_R[n]} \mathbb{R}_{\varphi} \arrow[u, "="]
\end{tikzcd}$$

The action of $\mathsf{fGC}_R[n]$ on $\mathrm{Tw}\,\mathsf{Gra}_R(U)$ is quasi-free with a good filtration, thus the functor $\mathrm{Tw}\,\mathsf{Gra}_R(U) \otimes_{\mathsf{fGC}_R[n]} (-)$ preserves quasi-isomorphisms. The two maps $d_0, d_1 : \Omega^*(\Delta^1) \to \mathbb{R}$ are quasi-isomorphisms, therefore all the maps in the diagram are quasi-isomorphisms.

If $\chi(M) = 0$, the proof that Graphs'_R and $\mathsf{Graphs}^{\varepsilon}_R$ assemble to Graphs_n-comodules is identical to the proof for $\mathsf{Graphs}^{\varphi}_R$ (see Proposition 2.3.31). It's also clear that the two zigzags define morphisms of comodules: in Graphs_n, all internal components are identified with zero anyway. □

Proposition 2.4.13. *The CDGA morphisms $\rho'_* : \mathsf{Gra}_R(U) \to \mathsf{G}_A(U)$ extend to CDGA morphisms $\rho_* : \mathsf{Graphs}^{\varepsilon}_R(U) \to \mathsf{G}_A(U)$ by sending all the graphs containing internal vertices to zero. If $\chi(M) = 0$ this extension defines a Hopf right comodule morphism.*

Proof. The submodule of graphs containing internal vertices is a multiplicative ideal and a cooperadic coideal, so all we are left to prove is that ρ_* is compatible with differentials. Since ρ'_* was a chain map, we must only prove that if Γ has internal vertices, then $\rho_*(d\Gamma) = 0$.

If a summand of $d\Gamma$ still contains an internal vertex, then it is mapped to zero by definition of ρ_*. The only parts of the differential that can remove all internal vertices are the contraction of dead ends and the splitting off of an internal component containing all the internal vertices of Γ. By the definition of $\mathsf{Graphs}^{\varepsilon}_R$, splitting off an internal component can yield a nonzero graph only if that internal component contains a single vertex, i.e. if we are dealing with a dead end.

Reusing the proof of Lemma 2.4.3, we can see that the contraction of that dead end cancels out with the splitting of that dead end after applying ρ_*. We thus get $\rho_*(d\Gamma) = 0$ as expected. □

2.4.2 The morphisms are quasi-isomorphisms

In this section we prove that the morphisms constructed in Proposition 2.3.31 and Proposition 2.4.13 are quasi-isomorphisms, completing the proof of Theorem C.

Theorem 2.4.14 (Precise version of Theorem C)**.** *The following zigzag, where the maps were constructed in Propositions 2.3.31, 2.4.12, and 2.4.13, is a zigzag of quasi-isomorphisms of CDGAs for all finite sets U:*

$$\mathsf{G}_A(U) \xleftarrow{\sim} \mathsf{Graphs}^{\varepsilon}_R(U) \xleftarrow{\sim} \mathsf{Graphs}'_R(U) \xrightarrow{\sim} \mathsf{Graphs}^{\varphi}_R(U) \xrightarrow{\sim} \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M(U)).$$

If $\chi(M) = 0$, the left-pointing maps form a quasi-isomorphism of Hopf right comodules:

$$(\mathsf{G}_A, \mathsf{e}_n^\vee) \xleftarrow{\sim} (\mathsf{Graphs}_R^\varepsilon, \mathsf{Graphs}_n) \xleftarrow{\sim} (\mathsf{Graphs}'_R, \mathsf{Graphs}_n).$$

If M is moreover framed, then the right-pointing maps also form a quasi-isomorphism of Hopf right comodules:

$$(\mathsf{Graphs}'_R, \mathsf{Graphs}_n) \xrightarrow{\sim} (\mathsf{Graphs}_R^\varphi, \mathsf{Graphs}_n) \xrightarrow{\sim} (\Omega^*_{\mathrm{PA}}(\mathsf{FM}_M), \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n)).$$

The rest of the section is dedicated to the proof of Theorem 2.4.14.

Remark 2.4.15. The graph complexes are, in general, nonzero even in negative degrees; see Remark 2.1.1.

Lemma 2.4.16. *The morphisms $\mathsf{Graphs}_R^\varepsilon(U) \to \mathsf{G}_A(U)$ factors through a quasi-isomorphism $\mathsf{Graphs}_R^\varepsilon(U) \to \mathsf{Graphs}_A(U)$, where $\mathsf{Graphs}_A(U)$ is the CDGA obtained by modding graphs with a label in $\ker(\rho : R \to A)$ in $\mathsf{Graphs}_R^\varepsilon(U)$.*

Proof. The morphism $\mathsf{Graphs}_R^\varepsilon \to \mathsf{Graphs}_A$ simply applies the surjective map $\rho : R \to A$ to all the labels. It is clear that $\mathsf{Graphs}_R^\varepsilon \to \mathsf{G}_A$ factors through the quotient.

We can consider the spectral sequences associated to the filtrations of both $\mathsf{Graphs}_R^\varepsilon$ and Graphs_A by the number of edges, and we obtain a morphism $\mathsf{E}^0\mathsf{Graphs}_R^\varepsilon \to \mathsf{E}^0\mathsf{Graphs}_A$. On both E^0 pages, only the internal differentials coming from R and A remain. The chain map $R \to A$ is a quasi-isomorphism, and so the morphism induces an isomorphism on the E^1 page. By standard spectral sequence arguments, it follows that $\mathsf{Graphs}_R^\varepsilon \to \mathsf{Graphs}_A$ is a quasi-isomorphism. □

The CDGA $\mathsf{Graphs}_A(U)$ has the same graphical description as $\mathsf{Graphs}_R^\varepsilon(U)$, except that now vertices are labeled by elements of A. An internal component with a single vertex labeled by $a \in A$ is identified with $\varepsilon(a)$, and an internal component with more than one vertex is identified with zero.

Proposition 2.4.17. *The morphism $\mathsf{Graphs}_A \to \mathsf{G}_A$ is a quasi-isomorphism.*

Before starting to prove this proposition, let us outline the different steps. We filter our complex in such a way that on the E^0 page, only the contracting part of the differential remains. Using a splitting result, we can focus on connected graphs. Finally, we use a "trick" (Figure 2.4.2) for moving labels around in a connected component, reducing ourselves to the case where only one vertex is labeled. We then get a chain map $A \otimes \mathsf{Graphs}_n \to A \otimes \mathsf{e}_n^\vee(U)$, which is a quasi-isomorphism thanks to the formality theorem.

Let us start with the first part of the outlined program, removing the splitting part of the differential from the picture. Define an increasing filtration on $\mathtt{Graphs}_A$ by counting the number of edges and vertices in a reduced graph:

$$F_s\mathtt{Graphs}_A = \{\Gamma \mid \#\text{edges} - \#\text{vertices} \leq s\}.$$

Lemma 2.4.18. *This is a filtration of chain complexes. It is bounded below for each finite set U: $F_{-\#U-1}\mathtt{Graphs}_A(U) = 0$.*

The E^0 page of the spectral sequence associated to this previous filtration is isomorphic as a module to $\mathtt{Graphs}_A$. Under this isomorphism the differential d^0 is equal to $d_A + d'_{\text{contr}}$, where d_A is the internal differential coming from A and d'_{contr} is the part of the differential that contracts all edges but dead ends.

Proof. If $\Gamma \in \mathtt{Graphs}_A(U)$ is the graph with no edges and no internal vertices, then it lives in filtration level $-\#U$. Adding edges can only increase the filtration. Since we consider reduced graphs (i.e. no internal components), each time we add an internal vertex (decreasing the filtration) we must add at least one edge (bringing it back up). By induction on the number of internal vertices, each graph is of filtration at least $-\#U$.

Let us now prove that the differential preserves the filtration and check which parts remain on the associated graded complex. The internal differential d_A doesn't change either the number of edges nor the number of vertices and so keeps the filtration constant. The contracting part d_{contr} of the differential decreases both by exactly one, and so keeps the filtration constant too.

The splitting part d_{split} of the differential removes one edge. If the resulting graph is still connected, then nothing else changes and this decreases the filtration by exactly 1. Otherwise, it means that a whole internal component γ was connected to the rest of the graph by a single edge, and then split off and identified with a number. If γ has a single vertex labeled by a (i.e. we split a dead end), then this number is $\varepsilon(a)$, and the filtration is kept constant. Otherwise, the summand is zero (and so the filtration is obviously preserved).

In all cases, the differential preserves the filtration, and so we get a filtered chain complex. On the associated graded complex, the only remaining parts of the differential are d_A, d_{contr}, and the part that splits off dead ends. But by the proof of Proposition 2.4.13 this last part cancels out with the part that contracts the dead ends. □

The symmetric algebra $S(\omega_{uv})_{u\neq v\in U}$ is weight graded, which induces a weight grading on $\mathsf{e}_n^\vee(U)$. This grading in turn induces an increasing filtration $F'_s\mathsf{G}_A$ on G_A (the extra differential strictly decreases the weight). Define a shifted filtration on G_A by:

$$F_s\mathsf{G}_A(U) = F'_{s+\#U}\mathsf{G}_A(U).$$

Lemma 2.4.19. *The* E^0 *page of the spectral sequence associated to* $F_*\mathsf{G}_A$ *is isomorphic as a module to* G_A. *Under this isomorphism the* d^0 *differential is just the internal differential of* A. □

Lemma 2.4.20. *The morphism* $\mathsf{Graphs}_A \to \mathsf{G}_A$ *preserves the filtration and induces a chain map, for each* U:

$$\mathsf{E}^0\mathsf{Graphs}_A(U) \to \mathsf{E}^0\mathsf{G}_A(U).$$

It maps graphs with internal vertices to zero, an edge e_{uv} *between external vertices to* ω_{uv}, *and a label* a *of an external vertex* u *to* $\iota_u(a)$.

Proof. The morphism $\mathsf{Graphs}_A(U) \to \mathsf{G}_A(U)$ preserves the filtration by construction.

If a graph has internal vertices, then its image in $\mathsf{G}_A(U)$ is of strictly lower filtration unless the graph is a forest (i.e. a product of trees). But trees have leaves, therefore by Corollary 2.3.34 and the formula defining $\mathsf{Graphs}_A \to \mathsf{G}_A$ they are mapped to zero in $\mathsf{G}_A(U)$ anyway. It's clear that the rest of the morphism preserves filtrations exactly, and so is given on the associated graded complex as stated in the lemma. □

For a partition π of U, define the submodule $\mathsf{Graphs}_A\langle\pi\rangle \subset \mathsf{E}^0\mathsf{Graphs}_A(U)$ spanned by reduced graphs Γ such that the partition of U induced by the connected components of Γ is exactly π. In particular let $\mathsf{Graphs}_A\langle U\rangle$ be the submodule of connected graphs.

Lemma 2.4.21. *For each partition* π *of* U, $\mathsf{Graphs}_A\langle\pi\rangle$ *is a subcomplex, and the complex* $\mathsf{E}^0\mathsf{Graphs}_A(U)$ *splits as the sum over all partitions* π *of* U:

$$\mathsf{E}^0\mathsf{Graphs}_A(U) = \bigoplus_{\pi} \bigotimes_{V\in\pi} \mathsf{Graphs}_A\langle V\rangle.$$

Proof. Since there is no longer any part of the differential that can split off connected components in $\mathsf{E}^0\mathsf{Graphs}_A$, it is clear that $\mathsf{Graphs}_A\langle U\rangle$ is a subcomplex. The splitting result is immediate. □

The complex $\mathsf{E}^0\mathsf{G}_A(U)$ splits in a similar fashion. For a monomial in $S(\omega_{uv})$, we say that u and v are "connected" if the term ω_{uv} appears in the monomial. Consider the equivalence relation generated by "u and v are connected". The monomial induces in this way a partition π of U, and this definition factors through the quotient defining $\mathsf{e}_n^\vee(U)$ (draw a picture of the 3-term relation). Finally, for a given monomial in $\mathsf{G}_A(U)$, the induced partition of U is still well-defined.

Thus for a given partition π of U, we can define $\mathsf{e}_n^\vee\langle\pi\rangle$ and $\mathsf{G}_A\langle\pi\rangle$ to be the submodules of $\mathsf{e}_n^\vee(U)$ and $\mathsf{E}^0\mathsf{G}_A(U)$ spanned by monomials inducing the partition

π. It is a standard fact that $\mathtt{e}_n^\vee\langle U\rangle = \mathtt{Lie}_n^\vee(U)$ [Sin07]. The proof of the following lemma is similar to the proof of the previous lemma:

Lemma 2.4.22. *For each partition π of U, $\mathtt{G}_A\langle\pi\rangle$ is a subcomplex of $\mathtt{E}^0\mathtt{G}_A(U)$, and there is a splitting as the sum over all partitions π of U:*

$$\mathtt{E}^0\mathtt{G}_A(U) = \bigoplus_{\pi} \bigotimes_{V\in\pi} \mathtt{G}_A\langle V\rangle. \qquad \square$$

Lemma 2.4.23. *The morphism $\mathtt{E}^0\mathtt{Graphs}_A(U) \to \mathtt{E}^0\mathtt{G}_A(U)$ preserves the splitting.* $\square$

We can now focus on connected graphs to prove Proposition 2.4.17.

Lemma 2.4.24. *The complex $\mathtt{G}_A\langle U\rangle$ is isomorphic to $A \otimes \mathtt{e}_n^\vee\langle U\rangle$.*

Proof. We define explicit isomorphisms in both directions.

Define $A^{\otimes U} \otimes \mathtt{e}_n^\vee\langle U\rangle \to A \otimes \mathtt{e}_n^\vee\langle U\rangle$ using the multiplication of A. This induces a map on the quotient $\mathtt{E}^0\mathtt{G}_A(U) \to A \otimes \mathtt{e}_n^\vee\langle U\rangle$, which restricts to a map $\mathtt{G}_A\langle U\rangle \to \mathtt{e}_n^\vee\langle U\rangle$. Since d_A is a derivation, this is a chain map.

Conversely, define $A \otimes \mathtt{e}_n^\vee\langle U\rangle \to A^{\otimes U} \otimes \mathtt{e}_n^\vee\langle U\rangle$ by $a \otimes x \mapsto \iota_u(a) \otimes x$ for some fixed $u \in U$ (it does not matter which one since $x \in \mathtt{e}_n^\vee\langle U\rangle$ is "connected"). This induces a map $A \otimes \mathtt{e}_n^\vee\langle U\rangle \to \mathtt{G}_A\langle U\rangle$, and it is straightforward to check that this map is the inverse isomorphism of the previous map. $\square$

We have a commutative diagram of complexes:

$$\begin{tikzcd}
\mathtt{Graphs}_A\langle U\rangle \arrow[r] \arrow[d] & A \otimes \mathtt{Graphs}'_n\langle U\rangle \arrow[d, "\sim"] \\
\mathtt{G}_A\langle U\rangle \arrow[r, "\cong"] & A \otimes \mathtt{e}_n^\vee\langle U\rangle
\end{tikzcd}$$

Here $\mathtt{Graphs}'_n(U)$ is defined similarly to $\mathtt{Graphs}_n(U)$ except that multiple edges are allowed. It is known that the quotient map $\mathtt{Graphs}'_n(U) \to \mathtt{e}_n^\vee(U)$ (which factors through $\mathtt{Graphs}_n(U)$) is a quasi-isomorphism [Wil14, Proposition 3.9]. The subcomplex $\mathtt{Graphs}'_n\langle U\rangle$ is spanned by connected graphs. The upper horizontal map in the diagram multiplies all the labels of a graph.

The right vertical map is the tensor product of id_A and $\mathtt{Graphs}_n\langle U\rangle \xrightarrow{\sim} \mathtt{e}_n^\vee\langle U\rangle$ (see 2.1.3). The bottom row is the isomorphism of the previous lemma.

It then remains to prove that $\mathtt{Graphs}_A\langle U\rangle \to A \otimes \mathtt{Graphs}'_n\langle U\rangle$ is a quasi-isomorphism to prove Proposition 2.4.17. If $U = \emptyset$, then $\mathtt{Graphs}'_A(\emptyset) = \mathbb{R} = \mathtt{G}_A(\emptyset)$ and the morphism is the identity, so there is nothing to do. From now on we assume that $\#U \geq 1$.

Lemma 2.4.25. *The morphism* $\mathtt{Graphs}_A\langle U\rangle \to A \otimes \mathtt{Graphs}'_n\langle U\rangle$ *is surjective on cohomology.*

Proof. Choose some $u \in U$. There is an explicit chain-level section of the morphism, sending $x \otimes \Gamma$ to $\Gamma_{u,x}$, the same graph with the vertex u labeled by x and all the other vertices labeled by 1_R. It is a well-defined chain map. It is clear that this is a section of the morphism in the lemma, hence the morphism of the lemma is surjective on cohomology. □

We now use a proof technique similar to the proof of [LV14, Lemma 8.3], working by induction. The dimension of $H^*(\mathtt{Graphs}'_n\langle U\rangle) = \mathtt{e}_n^\vee\langle U\rangle = \mathtt{Lie}_n^\vee(U)$ is well-known:

$$\dim H^i(\mathtt{Graphs}'_n\langle U\rangle) = \begin{cases} (\#U-1)!, & \text{if } i = (n-1)(\#U-1); \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.26}$$

Lemma 2.4.27. *For all sets* U *with* $\#U \geq 1$*, the dimension of* $H^i(\mathtt{Graphs}_A\langle U\rangle)$ *is the same as the dimension:*

$$\dim H^i(A \otimes \mathtt{Graphs}'_n\langle U\rangle) = (\#U-1)! \cdot \dim H^{i-(n-1)(\#U-1)}(A).$$

The proof will be by induction on the cardinality of U.

Lemma 2.4.28. *The complex* $\mathtt{Graphs}_A\langle \underline{1}\rangle$ *has the same cohomology as* A.

Proof. Let $\mathcal{J}$ be the subcomplex spanned by graphs with at least one internal vertex. We will show that $\mathcal{J}$ is acyclic; as $\mathtt{Graphs}_A\langle \underline{1}\rangle/\mathcal{J} \cong A$, this will prove the lemma.

There is an explicit homotopy h that shows that $\mathcal{J}$ is acyclic. Given a graph Γ with a single external vertex and at least one internal vertex, define $h(\Gamma)$ to be the same graph with the external vertex replaced by an internal vertex, a new external vertex labeled by 1_A, and an edge connecting the external vertex to the new internal vertex (see Figure 2.4.1).

The differential in $\mathtt{Graphs}_A\langle \underline{1}\rangle$ only retains the internal differential of A and the contracting part of the differential. Contracting the new edge in $h(\Gamma)$ gives back Γ, and it is now straightforward to check that:

$$dh(\Gamma) = \Gamma \pm h(d\Gamma).$$ □

Now let U be a set with at least two elements, and fix some element $u \in U$. Let $\mathtt{Graphs}_A^u\langle U\rangle \subset \mathtt{Graphs}_A\langle U\rangle$ be the subcomplex spanned by graphs Γ such that u has valence 1, is labeled by 1_A, and is connected to another external vertex.

We now get to the core of the proof. The idea (adapted from [LV14, Lemma 8.3]) is to "push" the labels of positive degree away from the chosen vertex u through a homotopy. Roughly speaking, we use Figure 2.4.2 to move labels around up to homotopy.

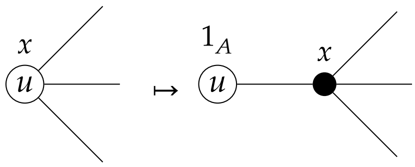


Figure 2.4.1: The homotopy h

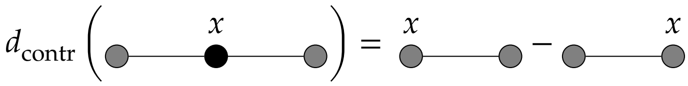


Figure 2.4.2: Trick for moving labels around (gray vertices are either internal or external)

Lemma 2.4.29. *The inclusion* $\mathtt{Graphs}_A^u\langle U\rangle \subset \mathtt{Graphs}_A\langle U\rangle$ *is a quasi-isomorphism.*

Proof. Let $\mathcal{Q}$ be the quotient. We will prove that it is acyclic. The module $\mathcal{Q}$ further decomposes into a direct sum of modules (but the differential does not preserve the direct sum):

- The module $\mathcal{Q}_1$ spanned by graphs where u is of valence 1, labeled by 1_A, and connected to an internal vertex;
- The module $\mathcal{Q}_2$ spanned by graphs where u is of valence ≥ 2 or has a label in $A^{>0}$.

We now filter $\mathcal{Q}$ as follows. For $s \in \mathbb{Z}$, let $F_s\mathcal{Q}_1$ be the submodule of $\mathcal{Q}_1$ spanned by graphs with at most $s+1$ edges, and let $F_s\mathcal{Q}_2$ be the submodule spanned by graphs with at most s edges. This filtration is preserved by the differential of $\mathcal{Q}$.

Consider the E^0 page of the spectral sequence associated to this filtration. Then the differential d^0 is an morphism $\mathsf{E}^0\mathcal{Q}_1 \to \mathsf{E}^0\mathcal{Q}_2$ by counting the number of edges (and using the crucial fact that dead ends are not contractible). It contracts the only edge incident to u. It is an isomorphism, with an inverse similar to the homotopy defined in Lemma 2.4.28, "blowing up" the point u into a new edge connecting u to a new internal vertex that replaces u.

This shows that $(\mathsf{E}^0\mathcal{Q}, d^0)$ is acyclic, hence $\mathsf{E}^1\mathcal{Q} = 0$. It follows that $\mathcal{Q}$ is acyclic. □

Proof of Lemma 2.4.27. The case $\#U = 0$ is obvious, and the case $\#U = 1$ of the lemma was covered in Lemma 2.4.28. We now work by induction and assumes the claim proved for $\#U \leq k$, for some $k \geq 1$.

Let U be of cardinality $k+1$. Choose some $u \in U$ and define $\mathtt{Graphs}^u_A\langle U\rangle$ as before. By Lemma 2.4.29 we only need to show that this complex has the right cohomology. It splits as:

$$\mathtt{Graphs}^u_A\langle U\rangle \cong \bigoplus_{v \in U - \{u\}} e_{uv} \cdot \mathtt{Graphs}_A\langle U - \{u\}\rangle \tag{2.4.30}$$

And therefore using the induction hypothesis:

$$\begin{aligned}\dim H^i(\mathtt{Graphs}^u_A\langle U\rangle) &= k \cdot \dim H^{i-(n-1)}(\mathtt{Graphs}_A\langle U - \{u\}\rangle) \\ &= k! \cdot \dim H^{i-k(n-1)}(A).\end{aligned}$$ □

Proof of Proposition 2.4.17. By Lemma 2.4.25, the morphism induced by $\mathtt{Graphs}_A \to \mathtt{G}_A$ on the E^0 page is surjective on cohomology. By Lemma 2.4.27 and Equation (2.4.26), both E^0 pages have the same cohomology, and so the induced morphism is a quasi-isomorphism. Standard spectral arguments imply the proposition. □

Proposition 2.4.31. *The morphism $\omega : \mathtt{Graphs}'_R(U) \to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M(U))$ is a quasi-isomorphism.*

Proof. From Equation (2.1.20), Proposition 2.4.12, Lemma 2.4.16, and Proposition 2.4.17, both CDGAs have the same cohomology of finite type, so it will suffice to show that the map is surjective on cohomology to prove that it is a quasi-isomorphism.

We work by induction. The case $U = \emptyset$ is immediate, as we get

$$\mathtt{Graphs}'_R(\emptyset) \xrightarrow{\sim} \mathtt{Graphs}^{\varphi}_R(\emptyset) = \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M(\emptyset)) = \mathbb{R}$$

and the last map is the identity.

Suppose that $U = \{u\}$ is a singleton. Since ρ is a quasi-isomorphism, for every cocycle $\alpha \in \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M(U)) = \Omega^*_{\mathrm{PA}}(M)$ there is some cocycle $x \in R$ such that $\rho(x)$ is cohomologous to α. Then the graph γ_x with a single (external) vertex labeled by x is a cocycle in $\mathtt{Graphs}'_R(U)$, and $\omega(\gamma_x) = \rho(x)$ is cohomologous to α. This proves that $\mathtt{Graphs}'_R(\{u\}) \to \Omega^*_{\mathrm{PA}}(M)$ is surjective on cohomology, and hence a quasi-isomorphism.

Now assume that $U = \{u\} \sqcup V$, where $\#V \geq 1$, and assume that the claim is proved for sets of vertices of size at most $\#V = \#U - 1$. By Equation (2.1.20), we may represent any cohomology class of $\mathtt{FM}_M(U)$ by an element $z \in \mathtt{G}_A(U)$ satisfying $dz = 0$. Using the relations defining $\mathtt{G}_A(U)$, we may write z as

$$z = z' + \sum_{v \in V} \omega_{uv} z_v,$$

where $z' \in A \otimes \mathtt{G}_A(V)$ and $z_v \in \mathtt{G}_A(V)$. The relation $dz = 0$ is equivalent to

$$dz' + \sum_{v \in V} (p_u \times p_v)^*(\Delta_A) \cdot z_v = 0, \tag{2.4.32}$$

$$\text{and } dz_v = 0 \text{ for all } v. \tag{2.4.33}$$

By the induction hypothesis, for all $v \in V$ there exists a cocycle $\gamma_v \in \mathtt{Graphs}'_R(V)$ such that $\omega(\gamma_v)$ represents the cohomology class of the cocycle z_v in $H^*(\mathtt{FM}_M(V))$, and such that $\sigma_*(\gamma_v)$ is equal to z_v up to a coboundary.

By Equation (2.4.32), the cocycle

$$\tilde{\gamma} = \sum_{v \in V} (p_u \times p_v)^*(\Delta_R) \cdot \gamma_v \in R \otimes \mathtt{Graphs}'_R(V)$$

is mapped to a coboundary in $A \otimes \mathtt{G}_A(V)$. The map $\sigma_* : R \otimes \mathtt{Graphs}'_R(V) \to A \otimes \mathtt{G}_A(V)$ is a quasi-isomorphism, hence $\tilde{\gamma} = d\tilde{\gamma}_1$ is a coboundary too.

It follows that $z' - \sigma_*(\tilde{\gamma}_1) \in A \otimes \mathtt{G}_A(V)$ is a cocycle. Thus by the induction hypothesis there exists some $\tilde{\gamma}_2 \in R \otimes \mathtt{Graphs}'_R(V)$ whose cohomology class represents the same cohomology class as $z' - \sigma_*(\tilde{\gamma}_1)$ in $H^*(A \otimes \mathtt{G}_A(V)) = H^*(M \times \mathtt{FM}_M(V))$.

We now let $\gamma' = -\tilde{\gamma}_1 + \tilde{\gamma}_2$, hence $d\gamma' = -\tilde{\gamma} + 0 = -\tilde{\gamma}$ and $\sigma_*(\gamma')$ is equal to z' up to a coboundary. By abuse of notation we still let γ' be the image of γ' under the obvious map $R \otimes \mathtt{Graphs}'_R(V) \to \mathtt{Graphs}'_R(U)$, $x \otimes \Gamma \mapsto \iota_u(x) \cdot \Gamma$. Then

$$\gamma = \gamma' + \sum_{v \in V} e_{uv} \cdot \gamma_v \in \mathtt{Graphs}'_R(U)$$

is a cocycle, and $\omega(\gamma)$ represents the cohomology class of z in $\Omega^*_{\mathrm{PA}}(\mathtt{FM}_M(U))$. We've shown that the morphism $\mathtt{Graphs}'_R(U) \to \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M(U))$ is surjective on cohomology, and hence it is a quasi-isomorphism. □

Proposition 2.4.34. *If A' is another Poincaré duality model of M, then we have a weak equivalence of symmetric collections $\mathtt{G}_A \simeq \mathtt{G}_{A'}$. If moreover $\chi(M) = 0$ then this weak equivalence is a weak equivalence of right Hopf $\mathtt{e}_n^\vee$-comodules.*

Proof. The CDGAs A and A' are models of the manifold, hence there exists some cofibrant CDGA S and quasi-isomorphisms $f : S \xrightarrow{\sim} A$ and $f' : S \xrightarrow{\sim} A'$. This yields two chain maps $\varepsilon, \varepsilon' : S \to \mathbb{R}[-n]$ defined by $\varepsilon = \varepsilon_A \circ f$ and $\varepsilon' = \varepsilon_{A'} \circ f'$. Mimicking the proof of Proposition 2.3.4, we can also find cocycles $\Delta, \Delta' \in S \otimes S$ which satisfy $\Delta^{21} = (-1)^n \Delta$ (and the same equation for Δ') and mapped respectively to Δ_A and $\Delta_{A'}$ under f and f'.

We can then build symmetric collections $\mathtt{Graphs}_S^{\varepsilon,\Delta}$ and $\mathtt{Graphs}_S^{\varepsilon',\Delta'}$ and quasi-isomorphisms $f_* : \mathtt{Graphs}_S^{\varepsilon,\Delta} \to \mathtt{G}_A$ and $f'_* : \mathtt{Graphs}_S^{\varepsilon',\Delta'} \to \mathtt{G}_{A'}$ like before. The differential of an edge e_{uv} in $\mathtt{Graphs}_S^{\varepsilon,\Delta}$ (resp. $\mathtt{Graphs}_S^{\varepsilon',\Delta'}$ is $\iota_{uv}(\Delta)$ (resp. $\iota_{uv}(\Delta')$), and an isolated internal vertex labeled by $x \in S$ is identified with $\varepsilon(x)$ (resp. $\varepsilon'(x)$).

If moreover $\chi(M) = 0$, we can choose Δ and Δ' such that their product vanishes, thus both $\mathtt{Graphs}_S$ become right Hopf $\mathtt{Graphs}_n$-comodules and the projections f_*, f'_* are compatible with the comodule structure. It thus suffices to prove that we have a quasi-isomorphism between $\mathtt{Graphs}_S^{\varepsilon,\Delta}$ and $\mathtt{Graphs}_S^{\varepsilon',\Delta'}$ to prove the proposition.

We first have an isomorphism $\mathtt{Graphs}_S^{\varepsilon',\Delta'} \cong \mathtt{Graphs}_S^{\varepsilon',\Delta}$ (with the obvious notations). Indeed, the two cocycles Δ and Δ' are cohomologous, say $\Delta' = \Delta - d\alpha$ for some $\alpha \in S \otimes S$ of degree $n-1$. If we replace α by $(\alpha + (-1)^n \alpha^{21})/2$, we can assume that $\alpha^{21} = (-1)^n \alpha$. Moreover if $\chi(M) = 0$, one can replace α by $\alpha - (\mu_S(\alpha) \otimes 1 + (-1)^n 1 \otimes \mu_S(\alpha))/2$ to be able to assume that $\mu_S(\alpha) = 0$. We then obtain an isomorphism by mapping an edge e_{uv} to $e_{uv} \pm \iota_{uv}(\alpha)$ (the sign depending on the direction of the isomorphism). This is clearly compatible with differentials, and if $\chi(M) = 0$ with the comodule structure.

The dg-module S is cofibrant and $\mathbb{R}[-n]$ is fibrant (like all dg-modules). Moreover we can assume that ε and ε' induce the same map on cohomology (it suffices to rescale, say, ε', and there is an automorphism of $\mathtt{Graphs}_S^{\varepsilon',\Delta}$ which takes care of this rescaling). Thus the two maps $\varepsilon, \varepsilon' : S \to \mathbb{R}[-n]$ are homotopic, and there exists some $h : S[1] \to \mathbb{R}[-n]$ such that $\varepsilon(x) - \varepsilon'(x) = h(dx)$ for all $x \in S$.

This homotopy induces a homotopy between the two morphisms $Z_\varepsilon, Z_{\varepsilon'} : \mathrm{fGC}_S[n] \to \mathbb{R}$. Because $\mathrm{Tw}\,\mathtt{Gra}_S^{\Delta}(U)$ and $\mathrm{Tw}\,\mathtt{Gra}_S^{\Delta'}(U)$ are cofibrant as modules over $\mathrm{fGC}_S[n]$, we obtain quasi-isomorphisms

$$\mathtt{Graphs}_S^{\varepsilon,\Delta} \simeq \mathtt{Graphs}_S^{\varepsilon',\Delta}$$

which are compatible with the comodule structure when $\chi(M) = 0$ (see the proof of Proposition 2.4.12). □

Proof of Theorem 2.4.14. The zigzag of the theorem becomes, after factorizing the first map through $\mathtt{Graphs}_A$:

$$\begin{tikzcd} \mathtt{Graphs}_A(U) \arrow[d] & \mathtt{Graphs}_R^{\varepsilon}(U) \arrow[l] & \mathtt{Graphs}'_R(U) \arrow[l] \arrow[r] & \mathtt{Graphs}_R^{\varepsilon}(U) \arrow[d] \\ \mathtt{G}_A(U) & & & \Omega^*_{\mathrm{PA}}(\mathtt{FM}_M(U)) \end{tikzcd}$$

All these maps are quasi-isomorphisms by Lemma 2.4.16, Proposition 2.4.12, Proposition 2.4.17, and Proposition 2.4.31. Their compatibility with the comodule structures (under the relevant hypotheses) are due to Proposition 2.3.31, Proposition 2.4.12, and Proposition 2.4.13.

The only thing missing from the theorem is that we must be able to choose any Poincaré duality model of the manifold, and not necessarily A; this is the content of Proposition 2.4.34. □

We can thus settle the conjecture of Lambrechts–Stanley over $\mathbb{R}$ for the class of manifolds that we consider:

Corollary 2.4.35. *Let M be a smooth simply connected closed manifold of dimension at least 4, and let A be any Poincaré duality model of M. Then the CDGA $\mathtt{G}_A(\underline{k})$ is a real model for $\mathrm{Conf}_k(M)$.* □

And this implies the real homotopy invariance of configuration spaces:

Corollary 2.4.36. *The real homotopy type of the configuration space of a smooth simply connected closed manifold of dimension at least 4 only depends on the real homotopy type of M.*

Proof. When $\dim M \geq 3$, the Fadell–Neuwirth fibrations [FN62] $\mathrm{Conf}_{k-1}(M - *) \hookrightarrow \mathrm{Conf}_k(M) \to M$ show by induction that if M is simply connected, then so is $\mathrm{Conf}_k(M)$ for all $k \geq 1$. Hence the real model $\mathtt{G}_A(\underline{k})$ completely encodes the real homotopy type of $\mathrm{Conf}_k(M)$. □

The degree-counting argument of Proposition 2.3.35 does not work in dimension 3, and so we cannot directly apply our results to S^3 (the only simply connected 3-manifold). We however record the following partial result, communicated to us by Thomas Willwacher:

Proposition 2.4.37. *The CDGA $\mathtt{G}_A(\underline{k})$, where $A = H^*(S^3; \mathbb{Q})$, is a rational model of $\mathrm{Conf}_k(S^3)$ for all $k \geq 0$.*

Proof. The claim is clear for $k = 0$. Since S^3 is a Lie group, the Fadell–Neuwirth fibration

$$\mathrm{Conf}_k(\mathbb{R}^3) \hookrightarrow \mathrm{Conf}_{k+1}(S^3) \to S^3$$

is trivial [FN62, Theorem 4]. The space $\mathrm{Conf}_{k+1}(S^3)$ is thus identified with $S^3 \times \mathrm{Conf}_k(\mathbb{R}^3)$, which is rationally formal with cohomology $H^*(S^3) \otimes \mathsf{e}_3^\vee(\underline{k})$. It thus suffices to build a quasi-isomorphism between $\mathsf{G}_A(\underline{k+1})$ and $H^*(S_3) \otimes \mathsf{e}_n^\vee(k)$ to prove the proposition.

To simplify notation, we consider $\mathsf{G}_A(\underline{k}_+)$ (where $\underline{k}_+ = \{0, \dots, k\}$), which is obviously isomorphic to $\mathsf{G}_A(\underline{k+1})$. Let us denote by $v \in H^3(S^3) = A^3$ the volume form of S^3, and recall that the diagonal class Δ_A is given by $1 \otimes v - v \otimes 1$. We have an explicit map given on generators by:

$$\begin{aligned} f : H^*(S^3) \otimes \mathsf{e}_3^\vee(\underline{k}) &\to \mathsf{G}_A(\underline{k}_+) \\ v \otimes 1 &\mapsto \iota_0(v) \\ 1 \otimes \omega_{ij} &\mapsto \omega_{ij} + \omega_{0i} - \omega_{0j} \end{aligned}$$

The Arnold relations show that this is a well-defined algebra morphism. Let us prove that $d \circ f = 0$ on the generator ω_{ij} (the vanishing on $v \otimes 1$ is clear). We may assume that $k = 2$ and $(i, j) = (1, 2)$, and then apply ι_{ij} to get the general case. Then we have:

$$\begin{aligned} df(\omega_{12}) &= (1 \otimes 1 \otimes v - 1 \otimes v \otimes 1) \\ &+ (1 \otimes v \otimes 1 - v \otimes 1 \otimes 1) \\ &- (1 \otimes 1 \otimes v - v \otimes 1 \otimes 1) \\ &= 0 \end{aligned}$$

We know that both CDGAs have the same cohomology (Equation (2.1.20)), so to check that f is a quasi-isomorphism it suffices to check that it is surjective in cohomology. The cohomology $H^*(\mathsf{G}_A(\underline{k}_+)) \cong H^*(S^3) \otimes \mathsf{e}_3^\vee(\underline{k})$ is generated in degrees 2 (by the ω_{ij}) and 3 (by the $\iota_i(v)$), so it suffices to check surjectivity in these degrees.

In degree 3, the cocycle $v \otimes 1$ is sent to a generator of $H^3(\mathsf{G}_A(\underline{k}_+)) \cong H^3(S^3) = \mathbb{Q}$. Indeed, assume $\iota_0(v) = d\omega$, where ω is a linear combination of the ω_{ij} for degree reasons. In $d\omega$, the sum of the coefficients of each $\iota_i(v)$ is zero, because they all come in pairs ($d\omega_{ij} = \iota_j(v) - \iota_i(v)$). We want the coefficient of $\iota_0(v)$ to be 1, so at least one of the other coefficient must be nonzero to compensate, hence $d\omega \neq \iota_0(v)$.

It remains to prove that $H^2(f)$ is surjective. We consider the quotient map $p : \mathsf{G}_A(\underline{k}_+) \to \mathsf{e}_3^\vee(\underline{k})$ that maps $\iota_i(v)$ and ω_{0i} to zero for all $1 \leq i \leq k$. We also consider the quotient map $q : H^*(S^3) \otimes \mathsf{e}_3^\vee(\underline{k}) \to \mathsf{e}_3^\vee(\underline{k})$ sending $v \otimes 1$ to zero. We get a morphism of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \ker q \arrow[r] \arrow[d] & H^*(S^3) \otimes \mathtt{e}_3^\vee(\underline{k}) \arrow[r, "q"] \arrow[d, "f"] & \mathtt{e}_3^\vee(\underline{k}) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & \ker p \arrow[r] & \mathtt{G}_A(\underline{k}) \arrow[r, "p"] & \mathtt{e}_3^\vee(\underline{k}) \arrow[r] & 0 \end{tikzcd}$$

We consider part of the long exact sequence in cohomology induced by these short exact sequences of complexes:

$$\begin{tikzcd} \mathtt{e}_3^\vee(\underline{k})^1 \arrow[r] \arrow[d, "="] & H^2(\ker q) \arrow[r] \arrow[d, "(1)"] & H^2(H^*(S^3) \otimes \mathtt{e}_3^\vee(\underline{k})) = \mathtt{e}_3^\vee(\underline{k})^2 \arrow[r] \arrow[d, "H^2(f)"] & \mathtt{e}_3^\vee(\underline{k})^2 \arrow[d, "="] \\ \mathtt{e}_3^\vee(\underline{k})^1 \arrow[r] & H^2(\ker p) \arrow[r] & H^2(\mathtt{G}_A(\underline{k}_+)) \arrow[r] & \mathtt{e}_3^\vee(\underline{k})^2 \end{tikzcd}$$

For degree reasons, $H^2(\ker q) = 0$ and so the map (1) is injective. By the four lemma, it follows that $H^2(f)$ is injective too. Both its domain and its codomain have the same finite dimension, thus $H^2(f)$ is an isomorphism. $\square$

2.5 Factorization homology of universal enveloping E_n-algebras

2.5.1 Factorization homology and formality

Fix some dimension n. Let U be a finite set and consider the space of framed embeddings of U copies of $\mathbb{R}^n$ in itself, endowed with the compact open topology:

$$\mathtt{Disk}_n^{\mathrm{fr}}(U) := \mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^n \times U, \mathbb{R}^n) \subset \mathrm{Map}(\mathbb{R}^n \times U, \mathbb{R}^n). \tag{2.5.1}$$

Using composition of embeddings, these spaces assemble to form a topological operad $\mathtt{Disk}_n^{\mathrm{fr}}$. This operad is weakly equivalent to the operad of little n-disks [AF15, Remark 2.10], and the application that takes $f \in \mathtt{Disk}_n^{\mathrm{fr}}(U)$ to $\{f(0 \times u)\}_{u \in U} \in \mathrm{Conf}_U(\mathbb{R}^n)$ is a homotopy equivalence.

Similarly if M is a framed manifold, then the spaces $\mathrm{Emb}^{\mathrm{fr}}(\mathbb{R}^n \times -, M)$ assemble to form a topological right $\mathtt{Disk}_n^{\mathrm{fr}}$-module, again given by composition of embeddings. We call it $\mathtt{Disk}_M^{\mathrm{fr}}$. If B is a $\mathtt{Disk}_n^{\mathrm{fr}}$-algebra, factorization homology is given by a derived composition product [AF15, Definition 3.2]:

$$\int_M B := \mathtt{Disk}_M^{\mathrm{fr}} \circ^{\mathbb{L}}_{\mathtt{Disk}_n^{\mathrm{fr}}} B = \mathrm{hocoeq}(\mathtt{Disk}_M^{\mathrm{fr}} \circ \mathtt{Disk}_n^{\mathrm{fr}} \circ B \rightrightarrows \mathtt{Disk}_M^{\mathrm{fr}} \circ B). \tag{2.5.2}$$

Thanks to [Tur13, Section 2], the pair $(\mathtt{FM}_M, \mathtt{FM}_n)$ is weakly equivalent to the pair $(\mathtt{Disk}^{\mathrm{fr}}_M, \mathtt{Disk}^{\mathrm{fr}}_n)$. So if B is an $\mathtt{FM}_n$-algebra, its factorization homology can be computed as:

$$\int_M B \simeq \mathtt{FM}_M \circ^{\mathbb{L}}_{\mathtt{FM}_n} B = \mathrm{hocoeq}(\mathtt{FM}_M \circ \mathtt{FM}_n \circ B \rightrightarrows \mathtt{FM}_M \circ B). \tag{2.5.3}$$

We now work in the category of chain complexes over $\mathbb{R}$. We use the formality theorem (Equation (2.1.15)) and the fact that weak equivalences of operads induce Quillen equivalence between categories of right modules (resp. categories of algebras) [Fre09, Theorems 16.A and 16.B]. Thus, to any homotopy class $[B]$ of E_n-algebras in the category of chain complexes, there corresponds a homotopy class $[\tilde{B}]$ of $\mathtt{e}_n$-algebras (which is generally not easy to describe).

Using Theorem 2.4.14, a game of adjunctions [Fre09, Theorems 15.1.A and 15.2.A] shows that:

$$\int_M B \simeq \mathtt{G}^{\vee}_A \circ^{\mathbb{L}}_{\mathtt{e}_n} \tilde{B}, \tag{2.5.4}$$

where A is the Poincaré duality model of M mentioned in the theorem, and $\mathtt{G}^{\vee}_A$ is the right $\mathtt{e}_n$-module dual to $\mathtt{G}_A$. Note that we forget the Hopf structure of $\mathtt{G}_A$ in Equation 2.5.4.

2.5.2 Higher enveloping algebras

Consider the forgetful functor from nonunital E_n-algebras to homotopy Lie algebras. Knudsen [Knu16, Theorem A] constructs a left adjoint U_n to this forgetful functor, called the higher enveloping algebra functor. He also gives a way of computing factorization homology of higher enveloping algebras If $\mathfrak{g}$ is a Lie algebra, then so is $A \otimes \mathfrak{g}$ for any CDGA A. Then the factorization homology of $U_n(\mathfrak{g})$ on M is given by [Knu17, Theorem 3.16].:

$$\int_M U_n(\mathfrak{g}) \simeq \mathrm{C}^{\mathrm{CE}}_*(\mathrm{A}^{-*}_{\mathrm{PL}}(M) \otimes \mathfrak{g}) \tag{2.5.5}$$

where $\mathrm{C}^{\mathrm{CE}}_*$ is the Chevalley–Eilenberg complex and $\mathrm{A}^{-*}_{\mathrm{PL}}(M)$ is the CDGA of rational piecewise polynomial differential forms, with the usual grading reversed to obtain a homological grading.

Let n be at least 2. We use again the formality theorem and standard facts about Quillen equivalences induced by weak equivalences of operads. Consider the canonical resolution $\mathtt{hoLie}_n \xrightarrow{\sim} \mathtt{Lie}_n$, the standard inclusion $\mathtt{Lie}_n \subset \mathtt{e}_n$, and the maps of operads $\mathtt{Lie}_n \to \mathtt{Graphs}^{\vee}_n$ which sends the generator (the bracket) to the graph with two external vertex and an edge between the two (see the part about twisting in Section 2.1.3). We can find a morphism of operads $\mathtt{hoLie}_n \to C_*(\mathtt{FM}_n)$

which makes the following diagram commute (the square on the right only up to homotopy):

$$\begin{tikzcd}
\mathtt{e}_n \arrow[r, "\sim"] & \mathtt{Graphs}_n^\vee & C_*(\mathtt{FM}_n) \arrow[l, "\sim"'] \\
\mathtt{Lie}_n \arrow[u, hook] \arrow[r, "="] & \mathtt{Lie}_n \arrow[u] \arrow[ur, phantom, "(h)"] & \mathtt{hoLie}_n \arrow[l, "\sim"'] \arrow[u, "\exists"]
\end{tikzcd}$$

These morphisms of operads yield functors between the corresponding categories of algebras. The forgetful functor U_n in Knudsen's work corresponds (in an informal sense) to the right adjoint (obtained by a Kan extension) of the induced functor $C_*(\mathtt{FM}_n)$-Alg $\to$ $\mathtt{hoLie}$-Alg which is the composite of the functor induced by the morphism of operads and a shift in degree.

The inclusion $\mathtt{Lie}_n \subset \mathtt{e}_n$ induces a functor $\mathtt{e}_n$-Alg $\to$ $\mathtt{Lie}$-Alg that maps an $(n-1)$-Poisson algebra B to its underlying Lie algebra $B[1-n]$. Its left adjoint $\tilde{U}_n$ corresponds to U_n, thanks to the commutative diagram above; in other words, given a Lie algebra $\mathfrak{g}$, which we can also see as an $\mathtt{hoLie}$-algebra, we have $U_n(\mathfrak{g}) \simeq \tilde{U}_n(\mathfrak{g})$. It is given by a symmetric algebra $\tilde{U}_n(\mathfrak{g}) = S(\mathfrak{g}[n-1])$, with a shifted Lie bracket is defined using the Leibniz rule.

Proposition 2.5.6. *Let A be a Poincaré duality CDGA. Then we have a quasi-isomorphism of chain complexes:*

$$\mathtt{G}_A^\vee \circ_{\mathtt{e}_n}^{\mathbb{L}} S(\mathfrak{g}[n-1]) \xrightarrow{\sim} \mathrm{C}_*^{\mathrm{CE}}(A^{-*} \otimes \mathfrak{g}).$$

If A is a Poincaré duality model of M, we have $A \simeq \Omega^*_{\mathrm{PA}}(M) \simeq \mathrm{A}^*_{\mathrm{PL}}(M) \otimes_{\mathbb{Q}} \mathbb{R}$ [Har+11, Theorem 6.1]. It follows that the Chevalley–Eilenberg complex of the previous proposition is weakly equivalent to the Chevalley–Eilenberg complex of Equation (2.5.5). By Equation (2.5.3), the derived circle product over $\mathtt{e}_n$ computes the factorization homology of $U_n(\mathfrak{g})$ on M, and so we recover a variant of Knudsen's theorem (over the reals) for closed framed simply connected manifolds.

Let $\mathtt{I}$ be the unit of the composition product, defined by $\mathtt{I}(\underline{1}) = \mathbb{R}$ and $\mathtt{I}(U) = 0$ for $\#U \neq 1$. Let Λ be the suspension of operads, satisfying

$$\Lambda\mathsf{P} \circ (X[-1]) = (\mathsf{P} \circ X)[-1] = \mathtt{I}[-1] \circ (\mathsf{P} \circ X).$$

As as symmetric collection, $\Lambda\mathsf{P}$ is simply given by $\Lambda\mathsf{P} = \mathtt{I}[-1] \circ \mathsf{P} \circ \mathtt{I}[1]$. Recall that we let $\mathtt{Lie}_n = \Lambda^{1-n}\mathtt{Lie}$.

The symmetric collection

$$\mathtt{L}_n := \mathtt{Lie} \circ \mathtt{I}[1-n] = \mathtt{I}[1-n] \circ \mathtt{Lie}_n \tag{2.5.7}$$

is a $(\mathtt{Lie}, \mathtt{Lie}_n)$-bimodule, i.e. a $\mathtt{Lie}$-algebra in the category of $\mathtt{Lie}_n$-right modules. We have $\mathtt{L}_n(U) = (\mathtt{Lie}_n(U))[1-n]$. This bimodule satisfies, for any Lie algebra $\mathfrak{g}$,

$$\mathtt{L}_n \circ_{\mathtt{Lie}_n} \mathfrak{g}[n-1] \cong \mathfrak{g} \text{ as Lie algebras.} \tag{2.5.8}$$

We can view the CDGA A^{-*} as a symmetric collection concentrated in arity 0, and as such it is a commutative algebra in the category of symmetric collections. Thus the tensor product

$$A^{-*} \otimes \mathtt{L}_n = \{A^{-*} \otimes \mathtt{L}_n(k)\}_{k \geq 0}$$

becomes a $\mathtt{Lie}$-algebra in right $\mathtt{Lie}_n$-modules, where the right $\mathtt{Lie}_n$-module structure comes from $\mathtt{L}_n$ and the Lie algebra structure combines the Lie algebra structure of $\mathtt{L}_n$ and the CDGA structure of A^{-*}. Its Chevalley–Eilenberg complex $C_*^{\mathrm{CE}}(A^{-*} \otimes \mathtt{L}_n)$ is well-defined, and by functoriality of C_*^{CE}, it is a right $\mathtt{Lie}_n$-module.

The proof of the following lemma is essentially found (in a different language) in the work of Félix and Thomas [FT04, Section 2].

Lemma 2.5.9. *The right $\mathtt{Lie}_n$-modules $\mathtt{G}_A^\vee$ and $C_*^{\mathrm{CE}}(A^{-*} \otimes \mathtt{L}_n)$ are isomorphic.*

Proof. We will actually define a non-degenerate pairing

$$\langle -, - \rangle : \mathtt{G}_A(U) \otimes C_*^{\mathrm{CE}}(A^{-*} \otimes \mathtt{L}_n)(U) \to \mathbb{R}$$

for each finite set U, compatible with differentials and the right $\mathtt{Lie}_n$-(co)module structures. As both complexes are finite-dimensional in each degree, this is sufficient to prove that they are isomorphic.

Recall that the Chevalley–Eilenberg complex $C_*^{\mathrm{CE}}(\mathfrak{g})$ is given by the cofree cocommutative conilpotent coalgebra $S^c(\mathfrak{g}[-1])$, together with a differential induced by the Koszul duality morphism $\Lambda^{-1}\mathtt{Com}^\vee \to \mathtt{Lie}$. It follows that as a module, $C_*^{\mathrm{CE}}(A^{-*} \otimes \mathtt{L}_n)(U)$ is given by:

$$\begin{aligned} &\bigoplus_{r \geq 0} \Bigl(\bigoplus_{\pi \in \mathrm{Part}_r(U)} A^{-*} \otimes \mathtt{L}_n(U_1)[-1] \otimes \dots \otimes A^{-*} \otimes \mathtt{L}_n(U_r)[-1] \Bigr)^{\Sigma_r} \\ &= \bigoplus_{r \geq 0} \Bigl(\bigoplus_{\pi \in \mathrm{Part}_r(U)} (A^{n-*})^{\otimes r} \otimes \mathtt{Lie}_n(U_1) \otimes \dots \otimes \mathtt{Lie}_n(U_r) \Bigr)^{\Sigma_r} \end{aligned} \tag{2.5.10}$$

where the sums run over all partitions $\pi = \{U_1 \sqcup \dots \sqcup U_r\}$ of U and $A^{n-*} = A^{-*}[-n]$ (which is a CDGA, Poincaré dual to A).

Fix some $r \geq 0$ and some partition $\pi = \{U_1 \sqcup \dots \sqcup U_r\}$. We define a first pairing:

$$(A^{\otimes U} \otimes \mathtt{e}_n^\vee(U)) \otimes ((A^{n-*})^{\otimes r} \otimes \mathtt{Lie}_n(U_1) \otimes \dots \otimes \mathtt{Lie}_n(U_r)) \to \mathbb{R} \tag{2.5.11}$$

as follows:

- On the A factors, the pairing uses the Poincaré duality pairing ε_A. It is given by:

$$(a_u)_{u\in U} \otimes (a'_1 \otimes \dots \otimes a'_r) \mapsto \pm\varepsilon_A(a_{U_1}\cdot a'_1)\dots\varepsilon_A(a_{U_r}\cdot a'_r),$$

  where $a_{U_i} = \prod_{u\in U_i} a_u$.

- On the factor $\mathtt{e}_n^\vee(U)\otimes\bigotimes_{i=1}^r \mathtt{Lie}_n(U_i)$, it uses the duality pairing on $\mathtt{e}_n^\vee(U)\otimes \mathtt{e}_n(U)$ (recalling that $\mathtt{e}_n = \mathtt{Com}\circ\mathtt{Lie}_n$ so we can view $\bigotimes_{i=1}^r \mathtt{Lie}_n(U_i)$ as a submodule of $\mathtt{e}_n(U)$).

The pairing in Equation (2.5.11) is the product of the two pairings we just defined. It is extended linearly on all of $(A^{\otimes U}\otimes \mathtt{e}_n^\vee(U))\otimes C_*^{\mathrm{CE}}(A^{-*}\otimes \mathtt{L}_n)(U)$, and it factors through the quotient defining $\mathtt{G}_A(U)$ from $A^{\otimes U}\otimes\mathtt{e}_n^\vee(U)$.

To check the non-degeneracy of this pairing, recall the vector subspaces $\mathtt{G}_A\langle\pi\rangle$ of Lemma 2.4.22, which are well-defined even though they are not preserved by the differential if we do not consider the graded space $\mathrm{E}^0\mathtt{G}_A$. Fix some partition $\pi = \{U_1,\dots,U_r\}$ of U, then we have an isomorphism of vector spaces:

$$\mathtt{G}_A\langle\pi\rangle \cong A^{\otimes r}\otimes\mathtt{Lie}_n^\vee(U_1)\otimes\dots\otimes A\otimes\mathtt{Lie}_n^\vee(U_r).$$

It is clear that $\mathtt{G}_A\langle\pi\rangle$ is paired with the factor corresponding to π in Equation (2.5.10), using the Poincaré duality pairing of A and the pairing between $\mathtt{Lie}_n$ and its dual; and if two elements correspond to different partitions, then their pairing is equal to zero. Since both ε_A and the pairing between $\mathtt{Lie}_n$ and its dual are non-degenerate, the total pairing is non-degenerate.

The pairing is compatible with the $\mathtt{Lie}_n$-(co)module structures, i.e. the following diagram commutes (a relatively easy but notationally tedious check):

$$\begin{tikzcd}
\begin{matrix}\mathtt{G}_A(U) & \otimes & C_*^{\mathrm{CE}}(A^{-*}\otimes\mathtt{L}_n)(U/W) \\ & \otimes & \mathtt{Lie}_n(W)\end{matrix} \arrow[r, "1\otimes\circ_W"] \arrow[d, "\circ_W^\vee\otimes 1"] & \mathtt{G}_A(U)\otimes C_*^{\mathrm{CE}}(A^{-*}\otimes\mathtt{L}_n)(U) \arrow[d, "{\langle -,-\rangle}"] \\
\begin{matrix}\mathtt{G}_A(U/W) & \otimes & C_*^{\mathrm{CE}}(A^{-*}\otimes\mathtt{L}_n)(U/W) \\ \mathtt{Lie}_n^\vee(W) & \otimes & \mathtt{Lie}_n(W)\end{matrix} \arrow[r, "{\langle -,-\rangle \atop \langle -,-\rangle_{\mathtt{Lie}_n}}"] & \mathbb{R}
\end{tikzcd}$$

Finally, it is a straightforward but long computation that the pairing commutes with differentials (i.e. $\langle d(-),-\rangle = \pm\langle -,d(-)\rangle$). It follows directly from the fact that $\varepsilon_A(aa') = \sum_{(\Delta_A)}\pm\varepsilon_A(a\Delta'_A)\varepsilon_A(a'\Delta''_A)$, which in turns stems from the definition of Δ_A. □

Proof of Proposition 2.5.6. The operad e_n is given by the composition product $\mathsf{Com} \circ \mathsf{Lie}_n$ equipped with a distributive law that encodes the Leibniz rule. We get the following isomorphism (natural in $\mathfrak{g}$):

$$\begin{aligned} \mathsf{G}_A^\vee \circ_{\mathsf{e}_n} S(\mathfrak{g}[n-1]) &= \mathsf{G}_A^\vee \circ_{\mathsf{e}_n} (\mathsf{Com} \circ \mathfrak{g}[n-1]) \\ &\cong \mathsf{G}_A^\vee \circ_{\mathsf{e}_n} (\mathsf{e}_n \circ_{\mathsf{Lie}_n} \mathfrak{g}[n-1]) \\ &\cong \mathsf{G}_A^\vee \circ_{\mathsf{Lie}_n} \mathfrak{g}[n-1]. \end{aligned}$$

According to Lemma 2.5.9, the right Lie_n-module $\mathsf{G}_A^\vee$ is isomorphic to $\mathrm{C}_*^{\mathrm{CE}}(A^{-*} \otimes \mathsf{L}_n)$. The functoriality of $A^{-*} \otimes -$ and $\mathrm{C}_*^{\mathrm{CE}}(-)$, as well as Equation (2.5.8), imply that we have the following isomorphism (natural in $\mathfrak{g}$):

$$\begin{aligned} \mathsf{G}_A^\vee \circ_{\mathsf{Lie}_n} \mathfrak{g}[n-1] &\cong \mathrm{C}_*^{\mathrm{CE}}(A^{-*} \otimes \mathsf{L}_n) \circ_{\mathsf{Lie}_n} \mathfrak{g}[n-1] \\ &\cong \mathrm{C}_*^{\mathrm{CE}}(A^{-*} \otimes ((\mathsf{L}_n) \circ_{\mathsf{Lie}_n} \mathfrak{g}[n-1])) \\ &\cong \mathrm{C}_*^{\mathrm{CE}}(A^{-*} \otimes \mathfrak{g}). \end{aligned}$$

The derived circle product is computed by taking a cofibrant resolution of $S(\mathfrak{g}[n-1])$. Let $Q_{\mathfrak{g}} \xrightarrow{\sim} \mathfrak{g}$ be a cofibrant resolution of the Lie algebra $\mathfrak{g}$. Then $S(Q_{\mathfrak{g}}[n-1])$ is a cofibrant e_n-algebra, and by Künneth's formula $S(Q_{\mathfrak{g}}[n-1]) \to S(\mathfrak{g}[n-1])$ is a quasi-isomorphism. It follows that:

$$\mathsf{G}_A^\vee \circ_{\mathsf{e}_n}^{\mathbb{L}} S(\mathfrak{g}[n-1]) = \mathsf{G}_A^\vee \circ_{\mathsf{e}_n} S(Q_{\mathfrak{g}}[n-1]).$$

We therefore have a commutative diagram:

$$\begin{tikzcd} \mathsf{G}_A^\vee \circ_{\mathsf{e}_n}^{\mathbb{L}} S(\mathfrak{g}[n-1]) \arrow[r] \arrow[d, "\cong"] & \mathsf{G}_A^\vee \circ_{\mathsf{e}_n} S(\mathfrak{g}[n-1]) \arrow[d, "\cong"] \\ \mathrm{C}_*^{\mathrm{CE}}(A^{-*} \otimes Q_{\mathfrak{g}}) \arrow[r] & \mathrm{C}_*^{\mathrm{CE}}(A^{-*} \otimes \mathfrak{g}) \end{tikzcd}$$

The Chevalley–Eilenberg functor preserves quasi-isomorphisms of Lie algebras, and therefore the bottom map is a quasi-isomorphism. The proposition follows. □

2.6 Outlook: The case of the 2-sphere and oriented manifolds

We provide a generalization of the previous work for the 2-sphere, and we formulate a conjecture for higher dimensional closed manifolds that are not necessarily framed.

2.6.1 Framed little disks and framed configurations

Following Salvatore–Wahl [SW03, Definition 2.1], we describe the framed little disks operad as a semi-direct product. If G is a topological group and $\mathtt{P}$ is an operad in G-spaces, the semi-direct product $\mathtt{P} \rtimes G$ is the topological operad defined by $(\mathtt{P} \rtimes G)(n) = \mathtt{P}(n) \times G^n$ and explicit formulas for the composition. Similarly if H is a commutative Hopf algebra and $\mathtt{C}$ is a Hopf cooperad in H-comodules, then the semi-direct product $\mathtt{C} \rtimes H$ is defined by formally dual formulas.

The operad $\mathtt{FM}_n$ is an operad in $\mathrm{SO}(n)$-spaces, the action rotating configurations. There is thus an operad $\mathtt{fFM}_n = \mathtt{FM}_n \rtimes \mathrm{SO}(n)$, the framed Fulton–MacPherson operad, weakly equivalent to the standard framed little disks operad.

Given an oriented n-manifold M, there is a corresponding right module over $\mathtt{fFM}_n$, which we call $\mathtt{fFM}_M$ [Tur13, Section 2]. The space $\mathtt{fFM}_M(U)$ is a principal $\mathrm{SO}(n)^{\times U}$-bundle over $\mathtt{FM}_M(U)$. Since $\mathrm{SO}(n)$ is an algebraic group, $\mathtt{fFM}_n$ and $\mathtt{fFM}_M(U)$ are respectively an operad and a module in semi-algebraic spaces.

2.6.2 Cohomology of $\mathtt{fFM}_n$ and potential model

The cohomology of $\mathrm{SO}(n)$ is classically given by Pontryagin classes and Euler classes:

$$\begin{aligned} H^*(\mathrm{SO}(2n);\mathbb{Q}) &= S(\beta_1, \dots, \beta_{n-1}, \alpha_{2n-1}) && (\deg \alpha_{2n-1} = 2n-1) \\ H^*(\mathrm{SO}(2n+1)) &= S(\beta_1, \dots, \beta_n) && (\deg \beta_i = 4i-1) \end{aligned}$$

By the Künneth formula, $\mathtt{fe}_n^\vee(U) = \mathtt{e}_n^\vee(U) \otimes H^*(\mathrm{SO}(n))^{\otimes U}$. We now provide explicit formulas for the cocomposition [SW03]. If $x \in H^*(\mathrm{SO}(n))$ and $u \in U$, denote as before $\iota_u(x) \in H^*(\mathrm{SO}(n))^{\otimes U}$. Let $W \subset U$, then if x is either β_i or α_{2n-1} in the even case, we have:

$$\circ_W^\vee(\iota_u(x)) = \begin{cases} \iota_*(x) \otimes 1 + 1 \otimes \iota_u(x), & \text{if } u \in W; \\ \iota_u(x) \otimes 1, & \text{otherwise.} \end{cases} \tag{2.6.1}$$

The formula for $\circ_W^\vee(\omega_{uv})$ depends on the parity of n. If n is odd, then $\circ_W^\vee(\omega_{uv})$ is still given by Equation (2.1.10). Otherwise, in $\mathtt{fe}_{2n}^\vee$ we have:

$$\circ_W^\vee(\omega_{uv}) = \begin{cases} \iota_*(\alpha_{2n-1}) \otimes 1 + 1 \otimes \omega_{uv}, & \text{if } u, v \in W; \\ \omega_{uv} \otimes 1, & \text{if } u, v \notin W; \\ \omega_{u*} \otimes 1, & \text{if } v \in W, u \notin W. \end{cases} \tag{2.6.2}$$

From now on, we focus on the case of oriented surfaces. Let $M = S^2$ be the 2-sphere, the only simply connected compact surface. We can choose $A =$

$H^*(S^2) = S(v)/(v^2)$ as our Poincaré duality model for S^2. The Euler class of A is $e_A = \chi(S^2)\mathrm{vol}_A = 2v$, and the diagonal class is given by $\Delta_A = v \otimes 1 + 1 \otimes v$. It is a standard fact about diagonal classes that $\mu_A(\Delta_A) = e_A$. Let also $\alpha \in H^1(S^1)$ be the generator (the "Euler class").

Definition 2.6.3. The **framed LS CDGA** $\mathtt{fG}_A(U)$ is given by:

$$\mathtt{fG}_A(U) = (A^{\otimes U} \otimes \mathtt{fe}_2^\vee(U)/(\iota_u(a) \cdot \omega_{uv} = \iota_v(a) \cdot \omega_{uv}), d),$$

where the differential is given by $d\omega_{uv} = \iota_{uv}(\Delta_A)$ and $d\iota_u(\alpha) = \iota_u(e_A)$.

Proposition 2.6.4. *The collection $\{\mathtt{fG}_A(U)\}_U$ is a Hopf right $\mathtt{fe}_2^\vee$-comodule, with cocomposition given by the same formula as Equation (2.2.2).*

Proof. The proofs that the cocomposition is compatible with the cooperad structure of $\mathtt{fe}_2^\vee$, and that this is compatible with the quotient, is the same as in the proof of Proposition 2.2.1. It remains to check compatibility with differentials.

We check this compatibility on generators. The internal differential of $A = H^*(S^2)$ is zero, so it is easy to check that $\circ_W^\vee(d(\iota_u(a))) = d(\circ_W^\vee(\iota_u(a))) = 0$. Similarly, using Equation (2.6.1), checking the equality on α is immediate.

As before there several are cases to check for ω_{uv}. If $u, v \in W$, then by Equation (2.6.2),

$$\begin{aligned} d(\circ_W^\vee(\omega_{uv})) &= d(\iota_*(\alpha) \otimes 1 + 1 \otimes \omega_{uv}) = \iota_*(e_A) \otimes 1 \\ &= \iota_*(\mu_A(\Delta_A)) \otimes 1 = \circ_W^\vee(d\omega_{uv}), \end{aligned}$$

and otherwise the proof is identical to the proof of Proposition 2.2.1. □

2.6.3 Connecting $\mathtt{fG}_A$ to $\Omega^*_{\mathrm{PA}}(\mathtt{fFM}_{S^2})$

The framed little 2-disks operad is known to be formal [GS10; Šev10]. We will focus on the proof of Giansiracusa–Salvatore [GS10], which goes along the same line as the proof of Kontsevich of the formality of $\mathtt{FM}_n$. To simplify notations, let $H = H^*(S^1)$, which is a Hopf algebra.

The operad $\mathtt{Graphs}_2$ is an operad in H-comodules, so we may consider the semi-direct product $\mathtt{Graphs}_2 \rtimes H$. Giansiracusa and Salvatore construct a zigzag:

$$\mathtt{fe}_2^\vee \xleftarrow{\sim} \mathtt{Graphs}_2 \rtimes H \xrightarrow{\sim} \Omega_{\mathrm{PA}}(\mathtt{fFM}_2). \tag{2.6.5}$$

The first map is the tensor product of $\mathtt{Graphs}_2 \xrightarrow{\sim} \mathtt{e}_2^\vee$ and the identity of H. The second map is given by the Kontsevich integral on $\mathtt{Graphs}_2$ and by sending the generator $\alpha \in H$ to the volume form of $\Omega^*_{\mathrm{PA}}(S^1)$ (pulled back by the relevant projection). They check that both maps are maps of Hopf (almost) cooperads, and they are quasi-isomorphisms by the Künneth formula.

Theorem 2.6.6. *The Hopf right comodule* $(\mathtt{fG}_A, \mathtt{fe}_2^\vee)$*, where* $A = H^*(S^2; \mathbb{R})$*, is quasi-isomorphic to the Hopf right comodule* $(\Omega^*_{\mathrm{PA}}(\mathtt{fFM}_{S^2}), \Omega^*_{\mathrm{PA}}(\mathtt{fFM}_2))$*.*

Proof. It is now straightforward to adapt the proof of Theorem C to this setting, reusing the proof of Giansiracusa–Salvatore [GS10]. We build the zigzag:

$$\mathtt{fG}_A \leftarrow \mathtt{Graphs}_A \rtimes H \to \Omega^*_{\mathrm{PA}}(\mathtt{fFM}_{S^2}).$$

We simply choose $R = A = H^*(S^2)$, mapping $v \in H^2(S^2)$ to the volume form of S^2. Note that the propagator can be made completely explicit on S^2, and it can be checked that Z_φ vanishes on all connected graphs with more than one vertex [CW16, Proposition 80]. The middle term is a Hopf right $(\mathtt{Graphs}_2 \rtimes H)$-comodule built out of $\mathtt{Graphs}_A$ and H, using formulas similar to the formulas defining $\mathtt{Graphs}_2 \rtimes H$ out of $\mathtt{Graphs}_2$ and H. The first map is given by the tensor product of $\mathtt{Graphs}_R \to \mathtt{G}_A$ and the identity of H.

The second map is given by the morphism of Proposition 2.3.31 on the $\mathtt{Graphs}_A$ factor, composed with the pullback along the projection $\mathtt{fFM}_{S^2} \to \mathtt{FM}_{S^2}$. The generator $\alpha \in H$ is sent to a pullback of a global angular form ψ of the principal $\mathrm{SO}(2)$-bundle $\mathtt{fFM}_{S^2}(\underline{1}) \to \mathtt{FM}_{S^2}(\underline{1}) = S^2$ induced by the orientation of S^2. This form satisfies $d\psi = \chi(S^2)\mathrm{vol}_{S^2}$.

The proof of Giansiracusa–Salvatore [GS10] then adapts itself to prove that these two maps are maps of Hopf right comodules. The Künneth formula implies that the first map is a quasi-isomorphism, and the second map induces an isomorphism on the E^2-page of the Serre spectral sequence associated to the bundle $\mathtt{fFM}_{S^2} \to \mathtt{FM}_{S^2}$ and hence is itself a quasi-isomorphism. □

Corollary 2.6.7. *The CDGA* $\mathtt{fG}_{H^*(S^2)}(\underline{k})$ *is a real model for* $\mathrm{Conf}^{\mathrm{or}}_k(S^2)$*, the principal* $\mathrm{SO}(2)^{\times k}$*-bundle over* $\mathrm{Conf}_k(S^2)$ *induced by the orientation of* S^2*.*

A conjecture in higher dimensions If M is an oriented n-manifold, Definition 2.6.3 readily adapts to define $\mathtt{fG}_{H^*(M)}$, by setting $d\alpha$ to be the Euler class of M (when n is even), and $d\beta_i$ to be the ith Pontryagin class of M. The proof of Proposition 2.6.4 adapts easily to this new setting, and $\mathtt{fG}_{H^*(M)}$ becomes a Hopf right $\mathtt{fe}_n^\vee$-comodule.

It was recently proved that the framed little disks operads is formal for even n and not formal for odd $n \geq 3$ [Mor16; KW17].

Conjecture 2.6.8. *If* M *is a formal, simply connected, oriented closed* $2n$*-manifold, then the pair* $(\mathtt{fG}_{H^*(M)}, \mathtt{fe}_{2n}^\vee)$ *is quasi-isomorphic to the pair* $(\Omega^*_{\mathrm{PA}}(\mathtt{fFM}_M), \Omega^*_{\mathrm{PA}}(\mathtt{fFM}_{2n}))$*.*

To adapt our proof directly to this conjecture, the difficulty would be the same as encountered by Giansiracusa–Salvatore [GS10], namely finding forms in

$\Omega^*_{\mathrm{PA}}(\mathtt{fFM}_n)$ corresponding to the generators of $H^*(\mathrm{SO}(n))$ and compatible with the Kontsevich integral. The proof of the formality of $\mathtt{fFM}_{2n}$ for $n > 1$ [KW17] is rather more involved than the proof for $n = 1$, and it would be an interesting question to try and adapt it for the conjecture.

If M itself is not formal, it is also not clear how to define Pontryagin classes in some Poincaré duality model of M – the Euler class was simply given the Euler characteristic. Nevertheless, for any oriented manifold M we get invariants of $\mathtt{fe}_n$-algebras by considering the functor $\mathtt{fG}_A^\vee \circ^{\mathbb{L}}_{\mathtt{fe}_n} (-)$. Despite not necessarily computing factorization homology, these invariants could prove to be interesting.

3 Configuration Spaces of Manifolds with Boundary

We now extend the results of the previous chapter to compact manifolds with boundary which admit a "surjective pretty model" [CLS15a].

For concreteness, let $(M, \partial M)$ be a compact manifold with boundary. Given some integer $k \geq 0$, its kth (ordered) configuration space is:

$$\mathrm{Conf}_k(M) := \{(x_1, \dots, x_k) \in M^{\times k} \mid x_i \neq x_j \ \forall i \neq j\}.$$

Suppose that the pair $(M, \partial M)$ has a surjective pretty model induced by $\psi : P \to Q$, where P is a Poincaré duality CDGA and ψ is surjective (see Section 3.1.2 for the definitions). In short, this means that M has been obtained by removing from a closed manifold N (of which P is a model) a sub-polyhedron X (of which Q is a model). Roughly, the pretty model construction is a way of encoding Poincaré–Lefschetz duality to obtain a model for $M = N - X$. This situation occurs the case if M is closed (in which case $Q = 0$), if both M and ∂M are 2-connected and the boundary retracts rationally onto its half-skeleton, if M is a disk bundle over a closed manifold, or if M is obtained by removing the thickening of a high-codimensional subpolyhedron from a closed manifold (see Theorem 3.1.12). Then a model of M is given by $A = P/I$, where $I = \mathrm{im}(\psi^!)$ is an ideal of the Poincaré duality CDGA P.

We use the same construction as in Chapter 2 to get a CDGA $\tilde{\mathsf{G}}_A(k)$ (Equation (3.3.19)), which is somehow a "perturbation" of $\mathsf{G}_A(k)$. We show that there is an isomorphism of graded vector spaces between $H^*(\mathsf{G}_A(k))$ and $H^*(\mathrm{Conf}_k(M))$ over $\mathbb{Q}$ (see Theorem 3.3.16), which generalizes the result of [LS08a]. If M is closed, then $I = 0$ and we recover the model considered in Chapter 2. If M is simply connected with simply connected boundary but does not admit a surjective pretty model, we have analogous results as soon as $\dim M \geq 7$ using what we dub "Poincaré–Lefschetz duality models".

Just like in the previous chapter, we consider a colored version of the Fulton–MacPherson compactification $\mathsf{SFM}_M(\emptyset, \underline{k})$. This is a stratified space which contains $\mathrm{Conf}_k(\mathring{M})$ as its interior, and the inclusion is a weak equivalence. If M is framed,

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Based on a work-in-progress in collaboration with P. Lambrechts.

then the collection $\mathsf{SFM}_M(\emptyset, -) = \{\mathsf{SFM}_M(\emptyset, \underline{k})\}_{k \geq 0}$ is equipped with an action of the Fulton–MacPherson operad, just like in the previous chapter.

The first result of this chapter can be summarized as:

Theorem D (See Theorem 3.5.37)**.** *Let M be a smooth, simply connected connected compact n-manifold with simply connected boundary of dimension at least 5. Assume that either M admits a surjective pretty model, or that $n \geq 7$ so that M admits a Poincaré–Lefschetz duality model. Let A be the model built either out of the surjective pretty model or the Poincaré–Lefschetz duality model.*

Then for all $k \geq 0$, the CDGA $\tilde{\mathsf{G}}_A(\underline{k})$ is weakly equivalent to $\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M(\emptyset, \underline{k}))$, and the equivalence is compatible with the action of the symmetric group Σ_k.

The collection $\tilde{\mathsf{G}}_A$ is a Hopf right comodule over the cooperad $\mathsf{e}_n^\vee = H^*(\mathsf{FM}_n)$. Moreover, if M is framed, then the right Hopf comodule $(\mathsf{G}_A, \mathsf{e}_n^\vee)$ is weakly equivalent to $(\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M(\emptyset, -)), \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n))$.

Corollary E. *Let M be a manifold with boundary which satisfies the hypotheses of Theorem D. Then the real homotopy type of $\mathrm{Conf}_k(M)$ only depends on the real homotopy type of the pair $(M, \partial M)$.*

One can also consider colored configuration in the manifold M and study the space, for $k, l \geq 0$:

$$\mathrm{Conf}_{k,l}(M) := \{(x_1, \dots, x_k, y_1, \dots, y_l) \in \mathrm{Conf}_{k+l}(M) \mid x_i \in \partial M, y_j \in \mathring{M}\}.$$

Once again, we can consider its (colored) Fulton–MacPherson compactification $\mathsf{SFM}_M(\underline{k}, \underline{l})$. This is a stratified space which contains $\mathrm{Conf}_{k,l}(M)$ as its interior, the inclusion being a homotopy equivalence. When M is framed, the collection SFM_M is equipped with an action of SFM_n, an operad weakly equivalent to the Swiss-Cheese operad [Vor99]. Willwacher [Wil15] has built a model $\mathsf{SGraphs}_n$ for SFM_n which is similar in spirit to Kontsevich's graph model Graphs_n (due to the non-formality of the Swiss-Cheese operad [Liv15], it is not possible to go all the way to the cohomology $H^*(\mathsf{SFM}_n)$). We then have:

Theorem F (See Theorem 3.5.42)**.** *Let M be a smooth, simply connected manifold with a simply connected boundary of dimension at least 5. Assume either that M admits a surjective pretty model or that $n \geq 7$. Then we have an explicit model $\mathsf{SGraphs}_R(\underline{k}, \underline{l})$ of $\mathrm{Conf}_{k,l}(M)$ which is built out of the pretty (resp. PLD) model. It is compatible with the action of the symmetric groups $\Sigma_k \times \Sigma_l$.*

The collection $\mathsf{SGraphs}_R$ is a Hopf right comodule over Willwacher's [Wil15] model $\mathsf{SGraphs}_n$ for the Swiss-Cheese operad. Moreover, if M is framed, then the Hopf right comodule $(\mathsf{SGraphs}_R, \mathsf{SGraphs}_n)$ is weakly equivalent to $(\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M), \Omega^*_{\mathrm{PA}}(\mathsf{SFM}_n))$.

Outline In Section 3.1, we recall some background on relative cooperads and comodules over them, pretty models, colored Fulton–MacPherson compactifications, and graph complexes. In Section 3.2, we define Poincaré–Lefschetz duality models, a generalization of surjective pretty models, and we prove that any simply connected manifold with simply connected boundary of dimension at least 7 admits a Poincaré–Lefschetz duality model. In Section 3.3, we describe the CDGA $\tilde{\mathsf{G}}_A(k)$, and we prove that it computes the Betti numbers of $\mathrm{Conf}_k(M)$. In Section 3.4, we compute the cohomology of the Swiss-cheese version of the graph complex twisted by a Maurer–Cartan element corresponding to the HKR isomorphism. In Section 3.5, we show that $\tilde{\mathsf{G}}_A(k)$ is actually a real model of $\mathrm{Conf}_k(M)$, and we prove that it is compatible with the action of the Fulton–MacPherson operad. We also build a graph complex $\mathsf{SGraphs}_R^{c_M, z_\varphi^S}(k,l)$ and we show that it is a real model for $\mathrm{Conf}_{k,l}(M)$, compatible with the action of the compactified colored Fulton–MacPherson operad.

3.1 Background and recollections

We mostly reuse the conventions of Section 2.1.1.

3.1.1 Colored (co)operads and (co)modules

We deal with special types of two-colored operads, called relative operads [Vor99], or Swiss-Cheese type operads.

Definition 3.1.1. Given an operad P, a **relative operad over** P is an operad in the category of right P-modules (see also Definition 1.1.1). This is equivalent to an operad with two colors (traditionally called the "closed" color $\mathfrak{c}$ and the "open" color $\mathfrak{o}$) such that operations with a closed output may only have closed inputs and are given by P.

We can encode the part of the operad with an open output as a **bisymmetric collection** Q, i.e. as a functor from the category of pairs of sets and pairs of bijections to dg-modules. The first set in the pair corresponds to open inputs, and the second to closed inputs. There is an identity $\eta_{\mathfrak{o}} \in \mathsf{Q}(\{*\}, \emptyset)$, and the operadic composition structure maps are given by:

$$\begin{aligned} \circ_T &: \mathsf{Q}(U, V/T) \otimes \mathsf{P}(T) \to \mathsf{Q}(U, V) && T \subset V; \\ \circ_{W,T} &: \mathsf{Q}(U/W, V) \otimes \mathsf{Q}(W, T) \to \mathsf{Q}(U, V \sqcup T) && W \subset U. \end{aligned}$$

As mentioned in the definition, we can equivalently view Q as an operad in the category of right P-modules. The P-module in arity U is given by $\mathsf{Q}(U, -)$, and one checks that the operad structure maps $\circ_{W,-} : \mathsf{Q}(U/W, -) \otimes \mathsf{Q}(W, -) \to \mathsf{Q}(U, -)$ are morphisms of right P-modules.

Given a (one-colored) cooperad C, a relative cooperad over C is defined dually as a bisymmetric collection D equipped with structure maps:

$$\circ_T^\vee : \mathsf{D}(U, V) \to \mathsf{D}(U, V/T) \otimes \mathsf{C}(W) \qquad T \subset V; \tag{3.1.2}$$

$$\circ_{W,T}^\vee : \mathsf{D}(U, V \sqcup T) \to \mathsf{D}(U/W, V) \otimes \mathsf{D}(W, T) \qquad W \subset U. \tag{3.1.3}$$

Finally, a comodule over a relative C-cooperad D is given by a bisymmetric collection N equipped with structure maps:

$$\circ_T^\vee : \mathsf{N}(U, V) \to \mathsf{N}(U, V/T) \otimes \mathsf{C}(T) \qquad T \subset V; \tag{3.1.4}$$

$$\circ_{W,T}^\vee : \mathsf{N}(U, V \sqcup T) \to \mathsf{N}(U/W, V) \otimes \mathsf{D}(W, T) \qquad W \subset U. \tag{3.1.5}$$

We can also define **relative Hopf cooperads** as relative cooperads in the category of CDGAs.

3.1.2 Pretty models for compact manifolds with boundary

The cohomology of a compact manifold with boundary does not satisfy Poincaré duality; it satisfies Poincaré–Lefschetz duality instead. Cordova Bulens, Lambrechts, and Stanley [CLS15a; CLS15b] use this idea to define "pretty rational models" for Poincaré duality pairs such as $(M, \partial M)$.

The rough idea is that one can see M as the complement in some closed manifold N of the thickening of some subpolyhedron $K \subset N$. One then takes a Poincaré duality model P of N and a model Q of K, and formally "kills" the forms that are dual to homology classes on K to obtain a model of M.

To be explicit, the starting data in the definition of a pretty model is:

- a Poincaré duality CDGA P of formal dimension n (in the rough picture above, it would be a model for N);
- a connected CDGA Q satisfying $Q^{\geq n/2-1} = 0$ (representing a model for $K \subset N$);
- a morphism $\psi : P \to Q$ (representing the restriction $\Omega^*(N) \to \Omega^*(K)$).

The morphism ψ makes Q into a P-module. Moreover, the dual $P^\vee$ (resp. $Q^\vee$) is a P-module (resp. Q-module) by letting $(x \cdot f)(y) = \pm f(xy)$, and so is the desuspension $P^\vee[-n]$ (resp. $Q^\vee[-n]$).

The Poincaré duality structure on P induces an isomorphism of P-modules:

$$\theta_P : P \xrightarrow{\cong} P^\vee[-n] \tag{3.1.6}$$

given by $a \mapsto \varepsilon_P(a \cdot -)$. The "shriek map" $\psi^!$ is the morphism of P-modules defined by the composite map:

$$\psi^! : Q^\vee[-n] \xrightarrow{\psi^\vee[-n]} P^\vee[-n] \xrightarrow[\cong]{\theta_P^{-1}} P. \tag{3.1.7}$$

More concretely, given $\alpha \in Q^\vee[-n]$, $\psi^!(\alpha)$ is uniquely determined by:

$$\forall x \in P,\ \varepsilon_P(\psi^!(\alpha)x) = \alpha(\psi(x)). \tag{3.1.8}$$

One must further assume that $\psi\psi^!$ is "balanced", i.e. that for all $f, g \in Q^\vee[-n]$, one has $\psi\psi^!(f) \cdot g = f \cdot \psi\psi^!(g)$. Note that, under our assumptions, this is automatically the case by degree reasons (in fact, we have $\psi\psi^! = 0$ because of the hypothesis that $Q^{\geq n/2-1} = 0$).

Definition 3.1.9. The **mapping cone** of a chain map $f : X \to Y$, denoted either by $\operatorname{cone}(f)$ or $Y \oplus_f X[1]$, is given as a graded vector space by $Y \oplus X[1]$, and the differential is given by $d(y, x) = (d_Y y + f(x), d_X x)$.

Note that $\psi\psi^! = 0$ implies that $\operatorname{cone}(\psi\psi^!) = Q \oplus_{\psi\psi^!} Q^\vee[1-n] = Q \oplus Q^\vee[1-n]$ is simply given by the direct sum of Q and $Q^\vee[1-n]$. The balancedness assumption ensures that one can make the mapping cones $P \oplus_{\psi^!} Q^\vee[1-n]$ and $Q \oplus Q^\vee[1-n]$ into CDGAs, using the product of P (resp. Q) and the P-module (resp. Q-module) structure of $Q^\vee[1-n]$, and by letting the product of two elements of $Q^\vee[1-n]$ be zero. In other words, the product is defined by:

$$(x, \alpha) \cdot (y, \beta) := (xy, x \cdot \beta + \alpha \cdot y).$$

Definition 3.1.10. The **pretty model** associated to (P, Q, ψ) is the CDGA morphism:

$$\lambda := \psi \oplus \mathrm{id} : P \oplus_{\psi^!} Q^\vee[1-n] \to Q \oplus Q^\vee[1-n].$$

The pair $(M, \partial M)$ admits a *surjective* pretty model if there exists a surjective morphism $\psi : P \to Q$ as above such that there is a commutative diagram of CDGAs:

$$\begin{tikzcd}
B := P \oplus_{\psi^!} Q^\vee[1-n] \arrow[d, "\lambda:=\psi\oplus\mathrm{id}"] & R \arrow[l, "\sim", "f"'] \arrow[r, "\sim", "g"'] \arrow[d, "\rho"] & \Omega^*(M) \arrow[d, "\mathrm{res}"] \\
B_\partial := Q \oplus Q^\vee[1-n] & R_\partial \arrow[l, "\sim", "f_\partial"'] \arrow[r, "\sim", "g_\partial"'] & \Omega^*(\partial M)
\end{tikzcd} \tag{3.1.11}$$

with each row a zigzag of quasi-isomorphisms.

Theorem 3.1.12 (Cordova Bulens, Lambrechts, and Stanley [CLS15a; CLS15b]). *The pair $(M, \partial M)$ admits a surjective pretty model in the following cases:*

- *If both M and ∂M are 2-connected and the boundary ∂M retracts rationally on its half-skeleton [CLS15a, Definition 6.1];*
- *If M (resp. ∂M) is the associated disk bundle (resp. sphere bundle) of a vector bundle of even rank over a simply connected Poincaré duality space;*
- *If M is obtained by removing a tubular neighborhood of a 2-connected subpolyhedron K from a 2-connected closed manifold N satisfying $2\dim K + 3 \leq \dim N = \dim M$.*

By the results from Section 2.1.4, when M is simply connected and $\partial M = \emptyset$, one can take a Poincaré duality model P of M and $Q = 0$ to obtain a surjective pretty model of $(M, \emptyset)$.

The CDGA B_∂ is a Poincaré duality CDGA of formal dimension $n-1$, with augmentation $\varepsilon_{B_\partial} : B_\partial \to \Bbbk[1-n]$ given on

$$(Q \oplus Q^\vee[1-n])^{n-1} = (Q^0)^\vee$$

by evaluation on $1_Q \in Q^0$. In other words the volume form is given by $1_Q^\vee$. We will also write $\varepsilon_B : B \to \Bbbk[-n]$ for the linear map given by $\varepsilon_B(x, y) = \varepsilon_P(x)$.

Remark 3.1.13. The map ε_B is not a chain map unless $Q = 0$. Indeed if $Q \neq 0$, then we have $d(1_Q^\vee) = \psi^!(1_Q^\vee) = \mathrm{vol}_P$, and thus $\varepsilon_B \circ d \neq 0$. Instead, there is a Stokes-like formula:

$$\varepsilon_B(d(x, \alpha)) = \varepsilon_{B_\partial}(\psi(x), \alpha), \quad \forall x \in P,\ \alpha \in Q^\vee[1-n], \tag{3.1.14}$$

which means that $\varepsilon = (\varepsilon_B, \varepsilon_{B_\partial})$ defines a chain map $\varepsilon : \mathrm{cone}(\psi \oplus \mathrm{id}) \to \mathbb{R}[-n+1]$.

Let $I \subset P$ be the image of $\psi^!$, which is an ideal. Then when ψ is surjective, the projection $P \oplus_{\psi^!} Q^\vee[1-n] \to A := P/I$ is a quasi-isomorphism. Thus if $(M, \partial M)$ admits a surjective pretty model, then M admits a model which is a quotient of a Poincaré duality CDGA. It is an open conjecture whether this is the case for all compact manifolds with boundary.

There is an interpretation of Poincaré–Lefschetz duality in this case. The kernel $K := \ker \psi \subset P$ is equal to $\ker(\psi \oplus \mathrm{id})$, which is a model for

$$\Omega^*(M, \partial M) := \mathrm{hoker}(\Omega^*(M) \to \Omega^*(\partial M)).$$

Recall the isomorphism $\theta_P : P \to P^\vee[-n]$ from Equation (3.1.6). It induces dual isomorphisms:

$$\theta_P : P/I \xrightarrow{\cong} K^\vee[-n], \qquad \vartheta_P : K \xrightarrow{\cong} (P/I)^\vee[-n], \tag{3.1.15}$$

that are both induced by the non-degenerate pairing $(P/I) \otimes (\ker \psi) \to \mathbb{R}[-n]$, $x \otimes y \mapsto \varepsilon_P(xy)$.

Example 3.1.16. Consider the manifold $M = D^n$ with boundary $\partial M = S^{n-1}$. Intuitively, we can see M as a sphere with a (thick) point removed. Thus consider the Poincaré duality CDGA $P = H^*(S^n) = S(\mathrm{vol}_n)/(\mathrm{vol}_n^2)$ (with a generator of degree n), and let $Q = H^*(\mathrm{pt}) = \mathbb{R}$. Let also $\psi : P \to Q$ be the augmentation.

We get as a model for D^n the mapping cone $B = P \oplus_{\psi^!} Q^\vee[1-n]$, which is spanned by three generators 1_P, $1_Q^\vee$, and vol_n. All the nontrivial products vanish, and $d(1_Q^\vee) = \mathrm{vol}_n$. The quotient $A = P/I$ is isomorphic to $H^*(D^n) = \mathbb{R}$, and $B_\partial = Q \oplus Q^\vee[1-n]$ is isomorphic to $H^*(\partial D^n) = H^*(S^{n-1})$ – both manifolds being formal.

3.1.3 Compactified configuration spaces

Recall the Fulton–MacPherson operad FM_n from Section 2.1.2, defined using compactified configuration spaces. Recall also its right modules FM_M, defined when M is a closed framed manifold. These constructions generalize to manifolds with boundary. Given a manifold with boundary $(M, \partial M)$ and finite sets U, V, define the colored configuration spaces:

$$\mathrm{Conf}_{U,V}(M) := \{c \in \mathrm{Conf}_{U \sqcup V}(M) \mid c(U) \subset \partial M,\ c(V) \subset \mathring{M}\}. \tag{3.1.17}$$

If M is a compact manifold, then this configuration space can be compactified into $\mathsf{SFM}_M(U,V)$. This is a stratified manifold of dimension $(n-1) \cdot \#U + n \cdot \#V$. Note that $\mathsf{SFM}_M(\emptyset, \{*\}) \cong M$, as a point may be infinitesimally close to the boundary of M.

Similarly if $M = \mathbb{H}^n \subset \mathbb{R}^n$ is the upper half-space, then $\mathrm{Conf}_{U,V}(M)$, after modding it out by the group of translations preserving $\mathbb{H}^n$ and positive dilatations, can be compactified into a space $\mathsf{SFM}_n(U,V)$ [Vor99]. It is a stratified manifold of dimension $n \cdot \#V + (n-1) \cdot \#U - n$ as soon as $\#U + 2\#V \geq 2$, and it is reduced to a point otherwise. One has homeomorphisms $\mathsf{SFM}_n(U, \emptyset) \cong \mathsf{FM}_{n-1}(U)$ and $\mathsf{SFM}_n(\underline{1}, \underline{1}) \cong D^{n-1}$, and homotopy equivalences $\mathsf{SFM}_n(\emptyset, V) \simeq \mathsf{FM}_n(V)$.

In this way, one obtains a relative operad SFM_n over FM_n, weakly equivalent to the Swiss-Cheese operad [Vor99] where operations with no open inputs are allowed (see [HL12, Section 3] or Remark 1.1.2 for an explanation of the difference). If M is framed, then the collection SFM_M also assemble into a right SFM_n-module using insertion of configurations:

$$\circ_T : \mathsf{SFM}_M(U, V/T) \times \mathsf{FM}_n(T) \to \mathsf{SFM}_M(U,V) \qquad T \subset V; \tag{3.1.18}$$
$$\circ_{W,T} : \mathsf{SFM}_M(U/W, V) \times \mathsf{SFM}_n(W,T) \to \mathsf{SFM}_M(U, V \sqcup T) \qquad W \subset U. \tag{3.1.19}$$

The spaces $\mathsf{SFM}_n(U, V)$ are clearly semi-algebraic sets. Moreover, M itself can be endowed with a semi-algebraic structure. Indeed, it can be triangulated and is C^1-diffeomorphic to some simplicial complex in some $\mathbb{R}^N$. Each simplex is a semi-algebraic set, and there is a finite number of them (because M is compact), hence M itself is semi-algebraic. It follows that the spaces $\mathsf{SFM}_M(U, V)$ are semi-algebraic too. We can thus apply the theory of PA forms (see Section 2.1.2) and obtain relative "almost" cooperads and comodules $\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_n)$ and $\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M)$.

Moreover, the projections

$$p_{U,V} : \mathsf{SFM}_M(U \sqcup I, V \sqcup J) \to \mathsf{SFM}_M(U, V) \tag{3.1.20}$$

which forget some of the points are semi-algebraic bundles. Just like in Chapter 2, we follow the proof from [LV14, Section 5.9] step by step, and we work by induction on the number of points forgotten (because SA bundles can be composed, see [Har+11, Proposition 8.5]). If we forget a terrestrial point, then this essentially follows from the fact that the projections $\mathsf{FM}_{\partial M}(U \sqcup I) \to \mathsf{FM}_{\partial M}(U)$ are SA bundles. For aerial points, let us give an idea of the key point of the proof for the projection $p : \mathsf{SFM}_M(\emptyset, \underline{2}) \to \mathsf{SFM}_M(\emptyset, \underline{1})$ which forgets the point 2. Recall that $\mathsf{SFM}_M(\emptyset, \underline{1})$ is homeomorphic to M. The fiber of p over a point x of the interior of M is given by M with an open disk centered around x. The fiber over a point x of the boundary has three types of points: either point 2 is far away from point 1, or it is infinitesimally close to point 1. The set of configurations from the first case is given by M with a half-disk around point 1 removed. The set of configurations from the second case is given by the space $\mathsf{SFM}_n(\emptyset, \underline{2})$ which specify how the two points are infinitesimally arranged. The intersection of these two sets is given by $\mathsf{SFM}_n(\emptyset, \underline{1})$, i.e. point 2 is infinitesimally close to 1 but "almost" leaves it. In other words, the fiber is:

$$p^{-1}(x) = M - \{\text{half-disk around } x\} \cup_{\mathsf{SFM}_n(\emptyset, \underline{1})} \mathsf{SFM}_n(\emptyset, \underline{2}), \tag{3.1.21}$$

which is also homeomorphic to M with an open disk removed, see Figure 3.1.1.

3.1.4 Graphs and the Swiss-Cheese operad

The Swiss-Cheese operad is not formal [Liv15; Wil17], thus there can be no quasi-isomorphism between $H^*(\mathsf{SFM}_n)$ and $\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_n)$. Nevertheless, there is a model $\mathsf{SGraphs}_n$ of SFM_n, due to Willwacher [Wil15], which is similar in spirit to the cooperad Graphs_n. The cohomology $H^*(\mathsf{SFM}_n) \cong H^*(\mathsf{FM}_n) \otimes H^*(\mathsf{FM}_{n-1})$ splits as a Voronov product (see [Vor99] and Section 1.4.3), and $\mathsf{SGraphs}_n$ is a way of intertwining Graphs_n and Graphs_{n-1} in a way that corrects the lack of formality.

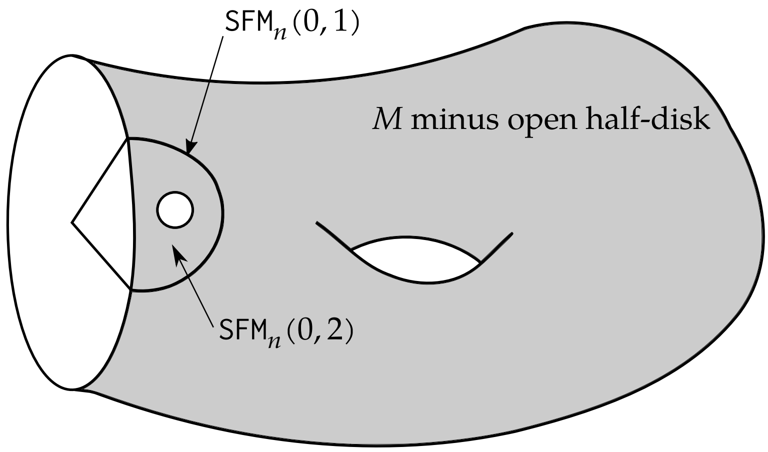


Figure 3.1.1: The fiber of $\mathsf{SFM}_M(\emptyset,2) \to \mathsf{SFM}_M(\emptyset,1) = M$ above a point of the boundary: $(M - \{\text{half-disk}\}) \cup_{\mathsf{SFM}_n(\emptyset,1)} \mathsf{SFM}_n(\emptyset,2)$

Remark 3.1.22. Our notations differ slightly from the notations of [Wil15]. We call $\mathsf{SGraphs}_n$ what is called Graphs_n^1 there, i.e. the space of operations with output of "type 1" (which corresponds to "open"). This is a relative Hopf cooperad over Graphs_n (which would be Graphs_n^2 in Willwacher's paper, the space of operations with output of "type 2", i.e. closed).

Let us assume from the start that $n \geq 3$ to avoid some difficulties that arise when $n = 2$. Note that in the $n = 2$ case we have found a different model (Chapter 1) for SFM_2 which is instead inspired by Tamarkin's proof of the formality of the little 2-disks operad, obtained by intertwining the operad of parenthesized permutations and the operad of parenthesized chord diagrams.

The idea is to construct a relative Hopf cooperad SGra_n over Gra_n, with two types of vertices: aerial ones, corresponding to closed inputs, and terrestrial ones, corresponding to open inputs. Edges are oriented, and the source of an edge may only be an aerial vertex. More concretely, one defines:

$$\mathsf{SGra}_n(U,V) := S(e_{vu})_{u \in U, v \in V} \otimes S(e_{vv'})_{v,v' \in V} \tag{3.1.23}$$

where the generators all have degree $n-1$ and the cooperad structure is given by formulas similar to Equation (2.1.10).

A monomial in $\mathsf{SGra}_n(U,V)$ can be seen as a directed graph with two kinds of vertices: aerial and terrestrial. The set U is the set of terrestrial vertices, and the set V is the set of aerial vertices. Note that unlike graphs in Gra_n, double edges (of the type e_{vu}^2 or $e_{vv'}^2$) and loops (also known as tadpoles, of the type e_{vv}) are allowed.

This allows us to produce a first morphism $\omega'' : \mathsf{SGra}_n \to \Omega^*_{\mathrm{PA}}(\mathsf{SFM}_n)$. One can define, for $v, v' \in V$, $\omega'(e_{vv'}) := p^*_{vv'}(\mathrm{vol}_{n-1})$, where

$$\mathrm{vol}_{n-1} \in \Omega^{n-1}_{\mathrm{PA}}(\mathtt{SFM}_n(\emptyset, \{v, v'\})) \simeq \Omega^{n-1}_{\mathrm{PA}}(\mathtt{FM}_n(\{v, v''\})) \simeq \Omega^{n-1}_{\mathrm{PA}}(S^{n-1}).$$

Recall that $\mathtt{SFM}_n(\{u\}, \{v\})$ is homeomorphic to D^{n-1}, and we write $\overline{\mathrm{vol}}^h_{n-1}$ for the $(n-1)$-form on $\mathtt{SFM}_n(\{u\}, \{v\})$ obtained by pulling back the volume form of S^{n-1} along the map $D^{n-1} \to S^{n-1}$ given by the hyperbolic geodesic (see [Wil15, Equation (8)]). Then for $u \in U$ and $v \in V$, define $\omega'(e_{vu}) := p^*_{vu}(\overline{\mathrm{vol}}^h_{n-1})$.

If $\Gamma \in \mathtt{SGra}_n(U, V)$ is a graph, let

$$c(\Gamma) := \int_{\mathtt{SFM}_n(U,V)} \omega'(\Gamma). \tag{3.1.24}$$

Note that these are analogous to the coefficients that appear in Kontsevich's universal L_∞ formality morphism $T_{\mathrm{poly}} \to D_{\mathrm{poly}}$, defined for $n = 2$ [Kon03].

The cooperad $\mathtt{SGra}_n$ is then twisted with respect to the sum of the Maurer–Cartan element $\mu \in \mathrm{Def}(\mathtt{hoLie}_n \to \mathtt{Gra}_n)$ with the Maurer–Cartan element defined by c, to obtain a relative Hopf cooperad $\mathrm{Tw}\,\mathtt{SGra}_n$ over $\mathrm{Tw}\,\mathtt{Gra}_n$.

Concretely, $\mathrm{Tw}\,\mathtt{SGra}_n(U, V)$ is spanned by graphs with four types of vertices: they can be either aerial or terrestrial, and either internal or external. Internal terrestrial vertices are of degree $1 - n$ and indistinguishable among themselves, and internal aerial vertices are of degree $-n$ and indistinguishable among themselves. Edges remain oriented and of degree $n - 1$. External terrestrial vertices are in bijection with U, and external aerial vertices are in bijection with V. The cooperad structure maps collapse subgraphs, and the product glues graphs along external vertices.

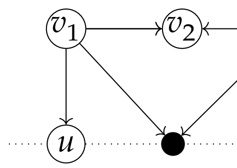


Figure 3.1.2: A colored graph in $\mathrm{Tw}\,\mathtt{SGra}_n(\{u\}, \{v_1, v_2\})$. Terrestrial vertices are drawn on a dotted line to distinguish them, even though they are not ordered. This graph is of degree $5(n-1) - n - (n-1)$.

The differential has several summands:

- A first summand (corresponding to μ) contracts edges between two aerial vertices, with at least one being internal.
- A second summand is given by contracting subgraphs $\Gamma' \subset \Gamma$ with at most one external vertex, which must be terrestrial, to obtain Γ/Γ', with

coefficient $c(\Gamma')$. One should note that in Γ/Γ', the new vertex representing the collapsed subgraph is terrestrial, even if Γ' is fully aerial.

- Finally, a third summand is given by forgetting some internal vertices and keeping a subgraph $\Gamma' \subset \Gamma$, with coefficient $c(\Gamma/\Gamma')$.

In the second and third cases, if the graph Γ/Γ' contains an edge whose source is terrestrial, then the summand is defined to be zero (see [Kon03, §6.4.2.2]).

One then checks that there is an extension $\omega : \mathrm{Tw}\,\mathtt{SGra}_n \to \Omega^*_{\mathrm{PA}}(\mathtt{SFM}_n)$, given by a formula analogous to Equation (2.1.14). It remains to mod out by the bi-ideal of graphs with internal components to obtain a Hopf cooperad $\mathtt{SGraphs}_n$, and to check that ω factors through the quotient.

Theorem 3.1.25 (Willwacher [Wil15]). *The morphism $\omega : \mathtt{SGraphs}_n \to \Omega^*_{\mathrm{PA}}(\mathtt{SFM}_n)$ is a quasi-isomorphism of relative Hopf cooperads.*

3.1.5 Graph complexes

There is a dg-module of particular interested appearing in Kontsevich's proof of the formality of $\mathtt{FM}_n$. Define

$$\mathrm{fGC}_n := \mathrm{Tw}\,\mathtt{Gra}_n(\emptyset)[-n] \tag{3.1.26}$$

to be the full graph complex. It is spanned by graphs containing only internal vertices, with a degree shift: if $\gamma \in \mathrm{fGC}_n$ is a graph with k edges and l vertices, then its degree $k(n-1) - ln + n$. The differential is given by contracting edges. Its suspension $\mathrm{fGC}_n[n]$ is a CDGA, and the product is the disjoint union of graphs. Since the differential cannot create new connected components, this a free CDGA, and we have:

$$\mathrm{fGC}_n = S(\mathrm{GC}_n[n])[-n] \tag{3.1.27}$$

where GC_n is the submodule of connected graph. The variants GC_n^2 (where bivalent vertices are allowed but univalent vertices forbidden) and $\mathrm{GC}_n^{\circlearrowleft}$ (where loops are allowed) are defined similarly.

Remark 3.1.28. Our notations are slightly nonstandard. It is often the dual complex $\mathrm{GC}_n^\vee$ which is called the graph complex, with a differential that is dually given by creating internal vertices. This dual is a Lie algebra using insertion of graphs.

The homology of this dg-module is particularly hard to compute. It is known that $H^0(\mathrm{GC}_2^\vee)$ is isomorphic to the Grothendieck–Teichmüller Lie algebra $\mathfrak{grt}_1$ and that $H^{<0}(\mathrm{GC}_2^\vee) = 0$ [Wil14]; it is a conjecture that $H^1(\mathrm{GC}_2) = 0$. The vanishing of $H^*(\mathrm{GC}_n^{2,\vee})$ in some degrees shows that the operad of little n-disks is intrinsically formal as a Hopf cooperad [FW15]. And this homology computes the homotopy groups of the space of rational automorphisms of the little n-disks operad [FTW17].

There is also a Swiss-Cheese version of that graph complex appearing in the description of Willwacher's model for the Swiss-Cheese operad:

$$\mathrm{fSGC}_n := \mathrm{Tw}\,\mathtt{SGra}_n(\emptyset,\emptyset)[-n] = S(\mathrm{SGC}_n[n])[-n] \tag{3.1.29}$$

is the full Swiss-Cheese graph complex, a symmetric algebra on its subcomplex of connected graphs. Its dual $\mathrm{SGC}_n^\vee$ is a Lie algebra using insertion of graphs (and the Lie algebra $\mathrm{GC}_n^\vee$ acts on SGC_n), and is sometimes also denoted KGC_n. The Kontsevich integrals $c \in \mathrm{SGC}_n^\vee$ define a Maurer–Cartan element in this Lie algebra.

3.2 Poincaré–Lefschetz duality models

3.2.1 Motivation

In Section 3.1.2, we recalled the definition of pretty models. If M admits a surjective pretty model (see Theorem 3.1.12) induced by $\psi : P \to Q$, then we get a diagram as in Equation (3.1.11):

$$\begin{tikzcd}
& K := \ker\lambda \arrow[dl, dashed, bend right, "\text{non degen.}"', "\text{pairing}"] \arrow[d, hook] & & \\
A := P/I & B := P \oplus_{\psi^!} Q^\vee[1-n] \arrow[l, two heads, "\sim", "\pi"'] \arrow[d, two heads, "\lambda=\psi\oplus\mathrm{id}"] & R \arrow[l, "\sim", "f"'] \arrow[r, "\sim", "g"'] \arrow[d, two heads, "\rho"] & \Omega^*_{\mathrm{PA}}(M) \arrow[d, "\mathrm{res}"] \\
& B_\partial := Q \oplus Q^\vee[1-n] & R_\partial \arrow[l, "\sim", "f_\partial"'] \arrow[r, "\sim", "g_\partial"'] & \Omega^*_{\mathrm{PA}}(\partial M)
\end{tikzcd} \tag{3.2.1}$$

Recall that $I \subset P$ is the image of $\psi^! : Q^\vee[1-n] \to P$. We consider the diagonal cocycle $\Delta_P \in P^{\otimes 2}$ as in Equation (2.1.18), and we implicitly view it as an element of $B^{\otimes 2}$ using the inclusion $B \subset P$. We let $\Delta_A \in A^{\otimes 2}$ be the image of Δ_P under the projection $\pi : B \to A$. Let K be the kernel of ψ, which is also the kernel of $\lambda = \psi \oplus \mathrm{id}$.

While they are quite useful, we do not know whether all manifolds with boundary admit a surjective pretty model. We will deal with a more general notion, which we dub "Poincaré–Lefschetz duality models" (or "PLD models" for short), and which are available as soon as $\dim M \geq 7$ and M and ∂M are simply connected. The goal is to obtain a diagram similar to Equation (3.2.1) without assuming that B (resp. B_∂) is of the form $P \oplus_{\psi^!} Q^\vee[1-n]$ (resp. $Q \oplus Q^\vee[1-n]$).

Let us first motivate our definition. It is always possible to find a surjective model ρ for M:[1]

$$\begin{tikzcd} R \arrow[r, "f", "\sim"'] \arrow[d, two heads, "\rho"] & \Omega^*_{\mathrm{PA}}(M) \arrow[d, "\mathrm{res}"] \\ R_\partial \arrow[r, "f_\partial", "\sim"'] & \Omega^*_{\mathrm{PA}}(\partial M) \end{tikzcd} \tag{3.2.2}$$

We can still define a chain map $(\varepsilon, \varepsilon_\partial) : \mathrm{cone}(\rho) \to \mathbb{R}[-n+1]$ by $\varepsilon(x) = \int_M f(x)$ and $\varepsilon_\partial(y) = \int_{\partial M} f_\partial(y)$. If we let $K_R = \ker(\rho)$, this yields dual maps $\theta_R : R \to K_R^\vee[-n]$ and $\vartheta_R : K_R \to R^\vee[-n]$, which are quasi-isomorphisms by Poincaré–Lefschetz duality. However, they are not necessarily isomorphisms, contrary to the maps θ_B, ϑ_B defined in the setting of surjective pretty models. This prevents us from carrying out the arguments of Section 3.3. The idea of this section is to adapt the proofs of [LS08b] in order to obtain a surjective model of $(M, \partial M)$ for which θ is surjective and ϑ is injective.

If the morphism θ_R were surjective (which is equivalent to ϑ_R being injective), then we would obtain an induced isomorphism $R/\ker\theta \cong K^\vee[-n]$, and the quotient map $R \to R/\ker\theta_R$ would be a quasi-isomorphism by the 2-out-of-3 property. We would thus get an actual isomorphism between a model of M and a model for the homology of M.

3.2.2 Definition and existence

Definition 3.2.3. A **Poincaré–Lefschetz duality pair** (or "**PLD pair**" for short) of formal dimension n is a CDGA morphism $\lambda : B \to B_\partial$ between two connected CDGAs, equipped with a chain map[2] $\varepsilon : \mathrm{cone}(\lambda) \to \mathbb{R}[1-n]$, and such that:

- The pair $(B_\partial, \varepsilon_{B_\partial})$ is a Poincaré duality CDGA of formal dimension $n-1$;
- Let $K := \ker\lambda$, and let $\theta_B : B \to K^\vee[-n]$ be defined by $\theta_B(b)(k) = \varepsilon(bk)$; then we require θ_B to be surjective and to be a quasi-isomorphism.

If $(B \xrightarrow{\lambda} B_\partial, \varepsilon)$ is a PLD pair, then

- the CDGA B is quasi-isomorphic to its quotient $A := B/\ker\theta_B$;
- the map $\theta : A \to K^\vee[-n]$ induced by θ_B is an isomorphism of $B^{\otimes 2}$-modules;

[1] We can also assume that the maps f, f_∂ factor through the sub-CDGAs of trivial forms in order to integrate along the fibers of the canonical projections, see [CW16, Appendix C].

[2] Recall that this is equivalent to the data of two linear maps $\varepsilon_B : B^n \to \mathbb{R}$ and $\varepsilon_{B_\partial} : B_\partial^{n-1} \to \mathbb{R}$ satisfying the "Stokes formula" $\varepsilon_B(dx) = \varepsilon_{B_\partial}(\lambda(x))$ and $\varepsilon_{B_\partial}(dy) = 0$.

- Equivalently, the pairing $A^i \otimes K^{n-i} \to \mathbb{R}$, given by $a \otimes k \mapsto \varepsilon_B(ak)$, is non-degenerate for all $i \in \mathbb{Z}$.

Example 3.2.4. A surjective pretty model is an example of a PLD pair.

Definition 3.2.5. A **Poincaré–Lefschetz duality model** of $(M, \partial M)$ is a PLD pair $\lambda : B \to B_\partial$ which is a model of the inclusion $\partial M \subset M$, in the sense that we can fill out the following diagram:

$$\begin{tikzcd} B \arrow[d, "\lambda"] & R \arrow[l, dashed, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\rho"] & \Omega^*_{\mathrm{PA}}(M) \arrow[d, "\mathrm{res}"] \\ B_\partial & R_\partial \arrow[l, dashed, "\sim"'] \arrow[r, "\sim"] & \Omega^*_{\mathrm{PA}}(\partial M) \end{tikzcd} .$$

Remark 3.2.6. In that case, $H^*(A) \cong H^*(M)$, $H^*(K) \cong H^*(M, \partial M)$, and the isomorphism θ is given by Poincaré–Lefschetz duality $H^*(M) \cong H_{n-*}(M, \partial M)$.

Proposition 3.2.7. *Let M be a simply connected n-manifold with simply connected boundary, and assume that $n \geq 7$. Then $(M, \partial M)$ admits a PLD model.*

Proof. We start with some surjective model $\rho : R \to R_\partial$ as in Equation (3.2.2), and we will build a PLD model out of it.

We keep the terminology of [LS08b]. We can find a surjective quasi-isomorphism $g_\partial : R_\partial \to B_\partial$, where B_∂ is a Poincaré duality CDGA, by [LS08b]. We let K_R be the kernel of $\rho''' := g_\partial \circ \rho$ and we define the chain map $\varepsilon_R : \mathrm{cone}(\rho') \to \mathbb{R}[-n+1]$. Let $\mathcal{O} := \ker \vartheta_R \subset K_R$ be the ideal of "orphans", i.e.

$$\mathcal{O} := \ker \vartheta_R = \{y \in K_R \mid \forall x \in R,\ \vartheta_R(y)(x) = \varepsilon_R(xy) = 0\}. \tag{3.2.8}$$

We could consider (for a moment) the new short exact sequence:

$$0 \to (K := K_R/\mathcal{O}) \to (B := R/\mathcal{O}) \xrightarrow{\lambda} B_\partial \to 0. \tag{3.2.9}$$

There is an induced chain map $\varepsilon_B : \mathrm{cone}(\lambda) \to \mathbb{R}[-n+1]$, and $\vartheta_B : K \to B^\vee[-n]$ is injective because we killed all the orphans. Thus we do obtain an isomorphism $B/\ker \theta_B \cong K^\vee[-n]$ induced by θ_B.

The problem is that the ideal $\mathcal{O}$ is not necessarily acyclic, thus λ is not necessarily a model of $(M, \partial M)$ anymore. Indeed, by Poincaré–Lefschetz duality, all we know is that a cycle $o \in \mathcal{O}$ is always the boundary of some element $z \in K$; but one may not always choose $z \in \mathcal{O}$. The idea, just like in [LS08b], is to extend the CDGA R by acyclic cofibrations (over B_∂) in order to make the ideal of orphans acyclic.

Thanks to our connectivity assumptions on the manifold and its boundary, we can assume that the model $\rho : R \to R_\partial$ of $(M, \partial M)$ and the chain map $\varepsilon_R : \mathrm{cone}(\rho') \to \mathbb{R}[-n+1]$ satisfy:

- ρ is surjective, hence so is $\rho' := g_\partial \circ \rho$, and we let $K_R = \ker(\rho')$;
- both R and R_∂ are of finite type;
- both R and R_∂ are simply connected, i.e. $R^0 = R^0_\partial = \mathbb{R}$ and $R^1 = R^1_\partial = 0$;
- we have $R^2_\partial \subset \ker d$ and $K^2 \subset \ker d$;[3]
- the morphisms θ_R, ϑ_R induced by ε_R are quasi-isomorphisms.

We say that the pair (ρ, ε_R) is a "good" pair if it satisfies all these assumptions. Let us say that its orphans are k-half-acyclic if $H^i(\mathcal{O}) = 0$ for $n/2 + 1 \leq i \leq k$. This condition is void when $k = n/2$, and if $k = n + 1$ then Poincaré–Lefschetz duality implies that $\mathcal{O}$ is acyclic (see [LS08b, Proposition 3.6]).

We will now work by induction (starting at $k = n/2$) and we assume that we are given a good pair (ρ, ε_R) whose orphans are $(k-1)$-half-acyclic. We will build an extension:

$$\begin{tikzcd}
0 \arrow[r] & K_R \arrow[r] \arrow[d, hook, "\sim"] & R \arrow[r, "\rho"] \arrow[d, hook, "\sim"] & R_\partial \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \hat{K}_R \arrow[r] & \hat{R} \arrow[r, "\hat{\rho}"] & R_\partial \arrow[r] & 0
\end{tikzcd} \tag{3.2.10}$$

and an extension $\hat{\varepsilon}_R : \operatorname{cone}(\hat{\rho}') \to \mathbb{R}[-n+1]$ of ε_R such that $(\hat{\rho}, \hat{\varepsilon})$ is good and its orphans are k-half-acyclic.

We follow closely [LS08b, Sections 4 and 5], adapting the proof where needed. Let $l = \dim(\mathcal{O}^k \cap \ker d) - \dim(d(\mathcal{O}^{k-1}))$ and choose l linearly independent cycles $\alpha_1, \dots, \alpha_l \in \mathcal{O}^k$ such that

$$\mathcal{O}^k \cap \ker d = d(\mathcal{O}^{k-1}) \oplus \langle \alpha_1, \dots, \alpha_l \rangle. \tag{3.2.11}$$

These are the obstructions to $\mathcal{O}$ being k-half acyclic. Because ϑ_R is a quasi-isomorphism and $\vartheta_R(\alpha_i) = 0$, there exists $\gamma_i' \in K^{k-1}$ such that $d\gamma_i' = \alpha_i$.

Let $m := \dim H^*(R) = \dim H^*(K_R)$, and choose cycles $h_1, \dots, h_m \in R$ such that $([h_1], \dots, [h_m])$ is a basis of $H^*(R)$. By Poincaré–Lefschetz duality there exists cycles $h_1', \dots, h_m'$ in K_R such that $\varepsilon(h_i h_j') = \delta_{ij}$ (and these form a basis for $H^*(K)$). Let

$$\gamma_i := \gamma_i' - \sum_j \varepsilon(\gamma_i' h_j) h_j' \in K^{k-1}, \tag{3.2.12}$$

[3] Since we require ρ to be surjective, it is possible that not all elements of R^2 are cycles, because some of the classes from $H^2(\partial M)$ may need to be killed. Check Example 3.1.16 when $M = D^3$.

and let Γ be the subspace of K^{k-1} generated by the γ_i. Then a proof similar to the proof of [LS08b, Lemma 4.1] shows that $d\gamma_i = \alpha_i$, and if $y \in R$ is a cycle, then $\varepsilon(\gamma_i y) = \vartheta(\gamma_i)(y) = 0$.

Now let

$$\hat{R} := (R \otimes S(c_1, \dots, c_l, w_1, \dots, w_l), dc_i = \alpha_i, dw_i = c_i - \gamma_i), \tag{3.2.13}$$

where the c_i and the w_i are new variables of degrees $k-1$ and $k-2$.

Extend ρ to $\hat{\rho} : \hat{R} \to R_\partial$ by declaring that $\hat{\rho}(c_i) = \hat{\rho}(w_i) = 0$. It follows that

$$\hat{K}_R := \ker \hat{\rho} = (K_R \otimes S(c_i, w_j)_{1 \le i,j \le l}, d). \tag{3.2.14}$$

It is not hard to see that $R \hookrightarrow \hat{R}$ is an acyclic cofibration (compare with [LS08b, Lemma 4.2]), and so is $K \hookrightarrow \hat{K}$. Finally, since all the γ_i and α_i are in $\ker \hat{\rho}$, one can extend ε_R to $\hat{\varepsilon}_R : \operatorname{cone}(\hat{\rho}) \to \mathbb{R}[-n+1]$ by formulas identical to [LS08b, Equation 4.5], which works because $n \ge 7$. It is clear that $(\hat{\rho}, \hat{\varepsilon}_R)$ is still a good pair. We let $\hat{\theta}_R$, $\hat{\vartheta}_R$ be the quasi-isomorphisms induced by the pairing, and we finally let $\hat{\mathcal{O}} := \ker \hat{\vartheta}_R$.

It remains to check that $\hat{\mathcal{O}}$ is k-half-acyclic, knowing that $\mathcal{O}$ is $(k-1)$-half-acyclic. First, we can reuse the proofs of [LS08b, Lemmas 5.2 and 5.3] to check that if $i > n-k+2$ then $\mathcal{O}^i \subseteq \hat{\mathcal{O}}^i$, and if $i \in \{k-2, k-1\}$ then $\hat{\mathcal{O}}^i \cap \ker d \subset \mathcal{O}^i \cap \ker d$. The only difference is that instead of the hypothesis $d(R^2) = 0$, we instead have $d(K_R^2) = 0$.

Now, just like [LS08b], we have several cases to check. If we have $k \ge n/2 + 2$, or if we have n odd and $k = (n+1)/2 + 1$, then the proof goes through unchanged up to slight changes of notation. However, if n is even and $k = n/2 + 1$, more significant adaptations are needed, even if the idea is the same. We'd like to check that $\hat{\mathcal{O}}^k \cap \ker d \subset d(\hat{\mathcal{O}}^{k-1})$. We already know that

$$\hat{\mathcal{O}}^k \cap \ker d \subset \mathcal{O}^k \cap \ker d = d(\mathcal{O}^{k-1}) \oplus \langle \alpha_i \rangle. \tag{3.2.15}$$

Since $\mathcal{O}^{k-1} \subset \hat{\mathcal{O}}^{k-1}$ (the proof is identical to [LS08b, Lemma 5.7]), it is sufficient to check that $\hat{\mathcal{O}}^{k-1} \cap \langle \alpha_i \rangle = 0$. Let us write $j : K_R \to R$ for the inclusion, $\bar{R} := R/\mathcal{O}$, $\bar{K}_R := K_R/\mathcal{O}$, $n = 2m$ and $k = n/2+1 = m+1$. Then we have long exact sequences in cohomology and morphisms between them:

$$\begin{tikzcd}
0 \arrow[r] & H^m(R) \arrow[r, "\pi"] & H^m(\bar{R}) \arrow[r, "\delta"] & H^{m+1}(\mathcal{O}) \arrow[r] & 0 \\
0 \arrow[r] & H^m(K) \arrow[r, "\pi"] \arrow[u, "j"] & H^m(\bar{K}_R) \arrow[r, "\delta"] \arrow[u, "\bar{j}"] & H^{m+1}(\mathcal{O}) \arrow[r] \arrow[u, "="] & 0
\end{tikzcd} \tag{3.2.16}$$

The space $H^{m+1}(\mathcal{O})$ is generated by the classes of the α_i. We obtain a section $\sigma : H^{m+1}(\mathcal{O}) \to H^m(\bar{K}_R)$ of δ by letting $\sigma([\alpha_i]) := [\gamma_i]$.

Suppose that we have some nonzero element $\alpha \in \langle \alpha_i \rangle$; we'd like to show that it is not in $\hat{\mathcal{O}}^k$, i.e. that it is not an orphan in $\hat{K}_R$. The pairing $H^m(\bar{K}_R) \otimes H^m(\bar{R}) \to \mathbb{R}$ induced by $\bar{\varepsilon}_R$ is non-degenerate, and $\sigma(\alpha) \neq 0$, hence there exists some $\beta \in H^m(\bar{R})$ such that $\varepsilon(\sigma(\alpha)\beta) \neq 0$. But as we saw before, for any $[h] = \sum x_j[h_j] \in H^m(R)$, $\varepsilon_R(\sigma(\alpha)h) = 0$, because $\varepsilon_R(\gamma_i h_j) = 0$. It follows that $\varepsilon_R(\sigma(\alpha)\sigma(\delta(\beta))) = \varepsilon_R(\sigma(\alpha)\beta) \neq 0$. If we write $\delta(\beta) = \sum_j \beta_j \gamma_j \in H^{m+1}(\mathcal{O})$, we can let $w = \sum_j \beta_j w_j \in \hat{R}$, and then by definition $\hat{\varepsilon}_R(\alpha w) = \varepsilon_R(\sigma(\alpha)\sigma(\delta(\beta))) \neq 0$, thus α is not an orphan. This completes the proof that $\hat{\mathcal{O}}^{k-1} \cap \langle \alpha_i \rangle = 0$, and thus that $\hat{\mathcal{O}}^k \cap \ker d \subset d(\hat{\mathcal{O}}^{k-1})$. We have thus covered all the cases to prove that if $\mathcal{O}$ is $(k-1)$-half-acyclic, then $\hat{\mathcal{O}}$ is k-half-acyclic.

By induction, we obtain a good pair $(\hat{\rho} : \hat{R} \to R_\partial, \hat{\varepsilon}_R)$ whose ideal of orphans is $(n+1)$-half acyclic (and hence actually acyclic), obtained by a sequence of acyclic cofibrations from R. It then remains to define $B := \hat{R}/\hat{\mathcal{O}}$ and ε_B to be the map induced by $\hat{\varepsilon}_R$ on the quotient to prove Proposition 3.2.7. $\square$

Given a PLD model of $(M, \partial M)$, we obtain a diagram (similar to (3.2.1)):

$$\begin{tikzcd}
 & K := \ker\lambda \arrow[ld, dashed, bend right, <->, "\text{non degen.}"', "\text{pairing}"] \arrow[d, hook] & & \\
A := B/\ker\theta_B & B \arrow[l, two heads, "\sim", "\pi"'] \arrow[d, two heads, "\lambda"] & R \arrow[l, "\sim", "g"'] \arrow[r, "\sim", "f"'] \arrow[d, two heads, "\rho"] & \Omega^*_{\mathrm{PA}}(M) \arrow[d, "\mathrm{res}"] \\
 & B_\partial & R_\partial \arrow[l, "\sim", "g_\partial"'] \arrow[r, "\sim", "f_\partial"'] & \Omega^*_{\mathrm{PA}}(\partial M)
\end{tikzcd} \tag{3.2.17}$$

We also see in the proof that ε_B satisfies by construction $\varepsilon_B g = \int_M f(-)$. We also have $\varepsilon_{B_\partial} g_\partial = \int_{\partial M} f_\partial(-)$ by construction (see Remark 2.1.17).

3.2.3 Diagonal classes

Let $(B \xrightarrow{\lambda} B_\partial, \varepsilon)$ be a Poincaré–Lefschetz duality pair, with $K = \ker\lambda$. Recall that we write $\theta : (A = B/\ker\theta_B) \to K^\vee[-n]$ for the isomorphism of $B^{\otimes 2}$-modules induced by the surjective $\theta_B : B \to K^\vee[-n]$.

The multiplication $\mu_K : K^{\otimes 2} \to K$ can then be dualized into a morphism of $B^{\otimes 2}$-modules $\delta : A \to A^{\otimes 2}[-n]$, and we let

$$\Delta_A := \delta(1_A) \in (A \otimes A)^n \tag{3.2.18}$$

be a representative of the diagonal class. Graded commutativity of μ_K implies that $\Delta_A^{21} = (-1)^n \Delta_A$. The fact that δ is a morphism of $B^{\otimes 2}$-modules imply that Δ_A satisfies

$$\forall a \in A,\ \delta(a) = (a \otimes 1)\Delta_A = \Delta_A(1 \otimes a). \tag{3.2.19}$$

Remark 3.2.20. If the PLD model comes from a surjective pretty model, then we check that this diagonal class is indeed the image of $\Delta_P \in P^{\otimes 2} \subset B^{\otimes 2}$ under the projection $\pi : B \to A$.

The duality between A and K can also be turned into a cocycle $\Delta_{AK} \in A \otimes K$. Let $\{a_i\}$ be some basis of A, and let $\{a_i^*\}$ be the dual basis of K, satisfying $\varepsilon_B(a_i a_i^*) = \delta_{ij}$. Then we define:

$$\Delta_{AK} := \sum \pm a_i \otimes a_i^* \in (A \otimes K)^n. \tag{3.2.21}$$

It is not hard to check that Δ_A is the image of Δ_{AK} under the composite map $1 \otimes \pi \circ \iota : A \otimes K \to A \otimes B \to A \otimes A$, and by definition:

$$\forall a \in A, \sum_{(\Delta_{AK})} \varepsilon_B(a\Delta''_{AK})\Delta'_{AK} = a. \tag{3.2.22}$$

Note also that, under the multiplication map $\mu_A : A \otimes A \to A$, we have $\mu_A(\Delta_A) = \sum \pm \pi(a_i a_i^*) = \chi(A)\pi(\mathrm{vol}_K)$ is equal to the Euler characteristic of B multiplied by the volume form $\mathrm{vol}_K \in K$ (the only element of B satisfying $\varepsilon_B(\mathrm{vol}_K) = 1$, representing the fundamental class of $(M, \partial M)$). But by degree reasons, $\mathrm{vol}_K \in K \subset B$ is in the kernel of $\theta_B : B \to K^\vee[-n]$ when $\partial M \neq \emptyset$, hence $\pi(\mathrm{vol}_K) = 0$ by definition. In other words,

$$\mu_A(\Delta_A) = 0 \text{ if } \partial M \neq \emptyset. \tag{3.2.23}$$

3.3 The model and its cohomology

From now on, let us assume that M is a smooth, simply connected manifold with nonempty boundary. We also fix a PLD model of M as in Equation (3.2.17).

3.3.1 The dg-module model of $\mathrm{Conf}_k(M)$

Given a finite set V and an element $v \in V$, define the canonical injection $\iota_v : A \to A^{\otimes V}$ by

$$\iota_v(a) := 1 \otimes \dots \otimes 1 \otimes \underbrace{a}_{v} \otimes 1 \otimes \dots \otimes 1. \tag{3.3.1}$$

Definition 3.3.2. Define a symmetric collection of CDGAs $\mathtt{G}_A$ by:

$$\mathtt{G}_A(V) := (A^{\otimes V} \otimes \mathtt{e}_n^\vee(V)/(\iota_v(a)\cdot\omega_{vv''} = \iota_{v'}(a)\cdot\omega_{vv'}),\ d(\omega_{vv'}) = (\iota_v \cdot \iota_{v'})(\Delta_A))$$

with the obvious actions of the symmetric groups.

This definition also makes sense when $\partial M = \emptyset$, and it yields the symmetric collection of CDGAs considered in Section 2.2.

When M is a closed manifold, as soon as its Euler characteristic $\chi(M)$ vanishes, then there is a structure of Hopf right $\mathsf{e}_n^\vee$-comodule on G_A (Proposition 2.2.1), with cocomposition structure maps characterized by (compare with Equation (2.1.10)):

$$\begin{aligned} \circ_T^\vee(\omega_{vv'}) &= 1 \otimes \omega_{vv'}, && \text{if } \{v, v'\} \subset T; \\ \circ_T^\vee(\omega_{vv'}) &= \omega_{[v][v']} \otimes 1, && \text{if } \{v, v'\} \not\subset T; \\ \circ_T^\vee(\iota_v(a)) &= \iota_{[v]}(a) && \text{for } a \in A,\ v \in V; \end{aligned} \tag{3.3.3}$$

where $[v] \in V/T$ is the class of v in the quotient.

Proposition 3.3.4. *If $\partial M \neq \emptyset$, then the symmetric collection of CDGAs G_A forms a right Hopf $\mathsf{e}_n^\vee$-comodule, with the same formulas.*

Proof. Comparing with the proof of Proposition 2.2.1, we see that almost all the arguments are the same. The only difficulty is to check that the cocomposition is compatible with the differential, which required that the Euler characteristic vanished in the boundaryless case.

It is immediate to check that $d(\circ_T^\vee(\iota_v(a))) = \circ_T^\vee(d(\iota_v(a))) = \iota_{[v]}(da)$, thus it suffices to check that the same equality holds on the generators $\omega_{vv'}$. If either $v \notin T$ or $v' \notin T$, this is again immediate; hence it suffices to check that

$$d(\circ_T^\vee(\omega_{vv'})) = \circ_T^\vee(d(\omega_{vv'})) \text{ for } v, v' \in T$$

The LHS of that equation always vanishes. On the other hand, the RHS is equal to $\iota_*(\mu_A(\Delta_A))$, where $\mu_A : A \otimes A \to A$ is the product. But by Equation (3.2.23), $\mu_A(\Delta_A) = 0$. $\square$

Example 3.3.5. Recall from Example 3.1.16 the model for (D^n, S^{n-1}), with $A = \mathbb{R}$, and $\Delta_A = 0$. It follows that in this case, G_A is isomorphic to $\mathsf{e}_n^\vee$ seen as a Hopf right comodule over itself. This is not surprising, given that FM_n is formal as an operad, and hence as a module over itself, and that $\mathsf{SFM}_{D^n}(\emptyset, -)$ is weakly equivalent to FM_n as a right FM_n-module.

3.3.2 Computing the homology

We now prove that G_A has the right cohomology, in the spirit of [LS08a] and using the methods of [CLS15b] to deal with manifolds with boundary. From then on and until the end of this section, we fix some integer $k \geq 0$. We can work over $\mathbb{Q}$ in this section.

The general idea goes as follows. If W is a manifold with boundary, and $X \subset W$ is a sub-polyhedron, then by [CLS15b], it suffices to know a CDGA model of the square of inclusions

$$\begin{tikzcd} \partial W \arrow[r, hook] & W \\ \partial_W X := X \cap \partial W \arrow[r, hook] \arrow[u, hook] & X \arrow[u, hook] \end{tikzcd} \tag{3.3.6}$$

to obtain a complex computing the cohomology of $W - X$.

Therefore, to compute the cohomology of $\mathrm{Conf}_k(M)$, we need to find such models for $W = M^k$ and $X = \Delta_{(k)} := \bigcup_{1 \le i,j \le k} \Delta_{ij}$, where $\Delta_{ij} := \{x \in M^k \mid x_i = x_j\}$. Since the sub-polyhedron $\Delta_{(k)}$ can be decomposed into the sub-polyhedra Δ_{ij}, we can use the techniques of [LS08a] to further simplify the description of the dg-module model as a "total cofiber" indexed by graphs, which will be isomorphic to G_A.

Let us now give the details. Let $E = \{(i,j) \mid 1 \le i < j \le k\}$ be a set of pairs, and let Γ be the poset of subsets of E ordered by reverse inclusion. We can see an element $\gamma \in \Gamma$ as a graph on k vertices, with an edge between i and j iff $(i,j) \in \gamma$. In particular $\emptyset \in \Gamma$ is the "empty" graph with no edges (but k vertices). Using this point of view, we can define the "zeroth homotopy group" $\pi_0(\gamma)$ of a graph $\gamma \in \Gamma$, which is a partition of $\{1, \dots, k\}$.

We obtain a functor ∇ from Γ to the category of topological spaces defined by

$$\gamma \mapsto \nabla(\gamma) := \bigcap_{e \in E_\gamma} \Delta_e \subset M^k, \tag{3.3.7}$$

where $\Delta_{(i,j)}$ is simply the small diagonal Δ_{ij}. Note that $\nabla(\emptyset) = M^k$, and that if $\gamma' \supset \gamma$ then there is an inclusion $\nabla(\gamma') \subset \nabla(\gamma)$. The space $\nabla(\gamma)$ is homeomorphic to the product $M^{\pi_0(\gamma)}$, and under these homeomorphisms, the inclusion $\nabla(\gamma') \subset \nabla(\gamma)$ is the cofibration induced by iterations of the diagonal map $M \to M \times M$. We thus obtain that:

$$\Delta_{(k)} = \bigcup_{1 \le i < j \le k} \Delta_{ij} = \mathrm{colim}_{\gamma \in \Gamma} \nabla(\gamma) = \mathrm{colim}_{\gamma \in \Gamma} M^{\pi_0(\gamma)}, \tag{3.3.8}$$

and this is in fact a homotopy colimit.

We will first aim to build a CDGA model for the square (3.3.6) (with $W = M^k$ and $X = \Delta_{(k)}$) out of the diagram (3.2.1). The previous description of $\Delta_{(k)}$ as a homotopy colimit tells us that a model for $\Delta_{(k)}$ is given by $\lim_{\gamma \in \Gamma^{\mathrm{op}}} B^{\otimes \pi_0(\gamma)}$, where the maps in the diagram are induced by iterations the multiplication μ_B of B. The inclusion $\Delta_{(k)}$ is modeled by the canonical map from $B^{\otimes k} = B^{\otimes \pi_0(\emptyset)}$ to the colimit.

It remains to find a model for $\partial_W X = \Delta_{(k)} \cap \partial(M^k)$ and models for the inclusion maps. The morphism $B \to B_\partial$ is surjective, hence B_∂ is isomorphic to B/K where $K := \ker(B \to B_\partial)$. We get:

Lemma 3.3.9. *For all $i \geq 0$, the left-hand side square is a CDGA model for the right-hand side square, where the horizontal maps are the diagonal maps:*

$$\begin{tikzcd}
B \arrow[d, two heads] & B^{\otimes i} \arrow[l, "\mu_B^{(i)}"'] \arrow[d, two heads] \\
B/K & B^{\otimes i}/K^{\otimes i} \arrow[l, "\mu_B^{(i)}"']
\end{tikzcd}
\quad \text{is a model for} \quad
\begin{tikzcd}
M \arrow[r, hook, "\delta"] & M^i \\
\partial M \arrow[u, hook] \arrow[r, hook, "\delta"] & \partial(M^i) \arrow[u, hook]
\end{tikzcd}$$

Proof. The idea is the same as in [CLS15b, Proposition 5.1]. We work by induction.

The case $i = 1$ is obvious (as $B_\partial \cong B/K$), and they prove the case $i = 2$. Now let use assume that the proposition is true for a given $i \geq 2$. There is a diagram, where all the inclusions are either induced by diagonal maps or induced by $\partial M \subset M$:

$$\begin{tikzcd}
M \times M^i & & M \arrow[ll, hook'] \\
\partial(M \times M^i) \arrow[u, hook] & M \times \partial(M^i) \arrow[l, hook'] & \\
(\partial M) \times M^i \arrow[u, hook] \arrow[ur, phantom, "\text{(ho. pushout)}"] & (\partial M) \times (\partial(M^i)) \arrow[l, hook'] \arrow[u, hook] & \partial M \arrow[l, hook'] \arrow[uu, hook]
\end{tikzcd} .$$

The diagram of the proposition is the "outer" diagram, and the bottom left square is a (homotopy) pushout.

Let P be the (homotopy) pullback in CDGAs of

$$B_\partial \otimes B^{\otimes i} \to B_\partial \otimes (B^{\otimes i}/K^{\otimes i}) \leftarrow B \otimes (B^{\otimes i}/K^{\otimes i}).$$

Then the induction hypothesis and the fact that homotopy pushouts of spaces become homotopy pullbacks of models imply that the following diagram is a CDGA model of the previous one (where the maps are either induced by $\mu_B : B^{\otimes 2} \to B$ or $\lambda : B \to B_\partial$):

$$\begin{tikzcd}
B \otimes B^{\otimes i} \arrow[rr] \arrow[d] & & B \arrow[dd] \\
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(ho. pullback)}"] & B \otimes (B^{\otimes i}/K^{\otimes i}) \arrow[d] & \\
B_\partial \otimes B^{\otimes i} \arrow[r] & B_\partial \otimes (B^{\otimes i}/K^{\otimes i}) \arrow[r] & B_\partial
\end{tikzcd}$$

Now, as in the proof [CLS15b, Lemma 5.3], it is clear that the natural map $B^{\otimes(i+1)} \to P$ is surjective and that its kernel is $K^{\otimes(i+1)}$, in other words that $P \cong B^{\otimes(i+1)}/K^{\otimes(i+1)}$. The proposition then follows immediately. □

The space $\Delta_{(k)} \cap \partial(M^k)$ admits a description as a colimit similar to the one Equation 3.3.8. Indeed, a point of M^k is in the boundary iff one of the coordinates is in the boundary of M. Now if a point is in both $\nabla(\gamma)$ and $\partial(M^k)$, then at least one of the coordinates (say x_i) is in the boundary, and thus all the points indexed by some j in the same connected component as i in γ is also in the boundary. We then obtain:

$$\Delta_{(k)} \cap \partial(M^k) = \operatorname{colim}_{\gamma\in\Gamma} \operatorname{colim}_{\emptyset \subsetneq S \subset \pi_0(\gamma)} (\partial M)^S \times M^{\pi_0(\gamma)-S}. \tag{3.3.10}$$

For a fixed γ, the inner colimit is precisely the image of $\partial(M^{\pi_0(\gamma)})$ under the diagonal embedding $M^{\pi_0(\gamma)} \hookrightarrow M^k$. Combining this with the previous lemma, we then obtain:

Proposition 3.3.11. *A model for the square (3.3.6), with $W = M^k$ and $X = \Delta_{(k)}$, is given by:*

$$\begin{tikzcd}
B^{\otimes k} \arrow[r, two heads, "\alpha_k"] \arrow[d, "\xi_k"] & B^{\otimes k}/K^{\otimes k} \arrow[d] \\
\lim_{\gamma\in\Gamma^{op}} R^{\otimes \pi_0(\gamma)} \arrow[r, "\beta_k"] & \lim_{\gamma\in\Gamma^{op}} R^{\otimes \pi_0(\gamma)}/K^{\otimes \pi_0(\gamma)}
\end{tikzcd}$$

□

The map α_k is surjective, thus we have a canonical quasi-isomorphism $\ker \alpha_k = K^{\otimes k} \xrightarrow{\sim} \operatorname{hoker} \alpha_k$. Therefore, by [CLS15b, Proposition 3.1]:

Corollary 3.3.12. *The cohomology of the cone*

$$\operatorname{cone}\left((\operatorname{hoker} \beta_k)^\vee[-nk] \xrightarrow{\bar{\xi}_k} (K^{\otimes k})^\vee[-nk]\right)$$

of the map induced by ξ_k on the kernels is isomorphic, as a graded vector space, to the cohomology of $\mathrm{Conf}_k(M) = M^k - \Delta_{(k)}$. □

Armed with this corollary, we can now define a cubical diagram $C_\bullet$ (see [LS08a, Section 7]) whose total cofiber computes the cohomology of $\mathrm{Conf}_k(M)$ as a graded vector space. Given $\gamma \in \Gamma$, the chain complex C_γ is defined to be $(K^{\otimes \pi_0(\gamma)})^\vee$; note in particular that $C_\emptyset = (K^{\otimes k})^\vee$. If $\gamma' \supset \gamma$, then the map $C_{\gamma'\supset\gamma} : C_{\gamma'} \to C_\gamma$ is induced by the dual of the multiplication of K.

Recall that the total cofiber of $C_\bullet$ is a representative of the homotopy colimit of C, given by the chain complex [LS08a, Definition 7.2]:

$$\operatorname{TotCof} C_\bullet := \Big(\bigoplus_{\gamma\in\Gamma} C_\gamma \cdot y_\gamma,\ D\Big), \tag{3.3.13}$$

where y_γ is some variable of degree $-\#E_\gamma$, $\deg(x\cdot y_\gamma) = \deg(x) + \deg(y_\gamma)$, and

$$D(x\cdot y_\gamma) = \pm(dx)\cdot y_\gamma + \sum_{e\in E_\gamma} \pm(C_{\gamma'\supset\gamma}x)\cdot y_{\gamma-e}. \tag{3.3.14}$$

Proposition 3.3.15. *The total cofiber of $C_\bullet$ computes the cohomology of $\operatorname{Conf}_k(M)$ as a graded vector space, up to suspension by nk.*

Proof. The proof is almost identical to the proof of [LS08a, Theorem 9.2]. First define an auxiliary cubical diagram $C'_\bullet$ just like $C_\bullet$, except for $C_\emptyset$ which we set equal to $(\operatorname{hoker}\beta_k)^\vee$. The maps $C'_{\gamma\supset\emptyset}$ are induced by the inclusions $K^{\otimes\pi_0(\gamma)} \to B^{\otimes\pi_0(\gamma)}$, which go through the definition of $\operatorname{hoker}\beta_k$. By Poincaré–Lefschetz duality, $(\operatorname{hoker}\beta_k)^\vee$ is a model for $\Delta_{(k)} = \bigcup_{\gamma\in\Gamma}\nabla(\gamma)$, each C_γ is a model for $\nabla(\gamma)$, and the maps between the C'_γ are models for the inclusions by Lemma 3.3.9. We thus obtain that $\operatorname{TotCof} C'_\bullet$ is acyclic by [LS08a, Proposition 9.1] and the homotopy invariance of total cofibers.

The morphism $C'_\gamma \to C_\gamma$ is given by the identity if $\gamma \neq \emptyset$, and it is given by $\bar{\xi}_k^\vee$ if $\gamma = \emptyset$. This yields a morphism of cubical diagrams $C'_\bullet \to C_\bullet$. Define $C''_\bullet$ to be the object-wise mapping cone of $C'_\bullet \to C_\bullet$, so that there is a short exact sequence

$$0 \to C'_\bullet \to C_\bullet \to C''_\bullet \to 0.$$

For $\gamma \neq \emptyset$, the map $C'_\gamma \to C_\gamma$ is the identity, hence C''_γ is acyclic. It follows that $\operatorname{TotCof} C''_\bullet$ is quasi-isomorphic to the cone of $\bar{\xi}_k^\vee : C'_\emptyset \to C_\emptyset$, which computes the cohomology of $\operatorname{Conf}_k(M)$ by Corollary 3.3.12.

There is a long exact sequence between the homologies of the total cofibers $C'_\bullet$, $C_\bullet$, and $C''_\bullet$: total cofibers commute with mapping cones up to homotopy, because both are types of homotopy colimits. The proposition then follows from the fact that $H_*(\operatorname{TotCof} C'_\bullet) = 0$. $\square$

Theorem 3.3.16. *Let M be a simply connected manifold with simply connected boundary and which admits a Poincaré–Lefschetz duality model. Let A be the model of M obtained from this PLD model (see Section 3.2). Then there is an isomorphism of graded vector spaces between $H^*(\mathsf{G}_A(k);\mathbb{Q})$ and $H^*(\operatorname{Conf}_k(M);\mathbb{Q})$.*

Proof. There is in fact an isomorphism of dg-modules

$$\mathtt{G}_A(k) \cong (\operatorname{TotCof} C_\bullet)[-nk].$$

It is induced by the Poincaré–Lefschetz isomorphism $\theta : A \to K^\vee[-n]$. We use the crucial fact that the multiplication of K is dual, under this isomorphism, to the map $A \to A^{\otimes 2}$ defined by $a \mapsto (a \otimes 1)\Delta_A = (1 \otimes a)\Delta_A$ (which is true by definition of Δ_A in our setting). The proof is then similar to the proof of Lemma 2.5.9. □

In other words, $\mathtt{G}_A(k)$ is a dg-module model of $\mathrm{Conf}_k(M)$. Unfortunately, in general if $\partial M \neq \emptyset$ then $\mathtt{G}_A(k)$ is *not* an actual model of $\mathrm{Conf}_k(M)$: the algebra structure is not the correct one.

3.3.3 The perturbed model

We define in the next section a "perturbed" version of $\mathtt{G}_A(V)$, which will be the actual model for $\mathrm{Conf}_k(M)$, and we will prove that it is isomorphic to $\mathtt{G}_A(V)$ as a dg-module.

Choose some section $s : B_\partial \to B$ of ρ. Using the Poincaré duality of B_∂, this section s corresponds to some element $\sigma_B \in B_\partial \otimes B$ of degree $n-1$. Note that, by definition, the image of σ_B in $B_\partial^{\otimes 2}$ is the diagonal class of B_∂.

The map s does not commute, in general, with the differential, though for any $x \in B_\partial$, one has $ds(x) - s(dx) \in K = \ker \rho$. It follows that in general $d\sigma_B \neq 0$, but $d\sigma_B \in B_\partial \otimes K$. We in fact have that $d\sigma_B$ is a representative of the image of Δ_{AK} under the map $B \otimes K \to B_\partial \otimes K$.

Let

$$\sigma_A = (1 \otimes \pi)(\sigma_B) := \sum_{(\sigma_A)} \sigma' \otimes \sigma'' \in B_\partial \otimes A. \tag{3.3.17}$$

Remark 3.3.18. We are going to use Sweedler's notation as above extensively. If the element σ (or any other tensor) appears multiple times in an equation, we are going to write it as follows:

$$\sigma \otimes \sigma = \sum \sigma_1' \otimes \sigma_1'' \otimes \sigma_2' \otimes \sigma_2''$$

and so on if it appears more than twice.

We define the perturbed model to be:

$$\tilde{\mathtt{G}}_A(V) := (A^{\otimes V} \otimes S(\tilde{\omega}_{vv'})_{v,v' \in V}/J, d\tilde{\omega}_{vv'} = (\iota_v \cdot \iota_{v'})(\Delta_A)) \tag{3.3.19}$$

where the ideal of relations is generated by $\tilde{\omega}_{vv'}^2 = 0$ and, for all $b \in B$ and all subsets $T \subset V$ of cardinality at least two:

$$\sum_{v \in T} \pm(\iota_v(\pi(b)) \cdot \prod_{v \neq v' \in T} \tilde{\omega}_{vv'}) + \sum \pm \varepsilon_\partial(\rho(b) \prod_{v \in T} \sigma_v') \prod_{v \in T} \iota_v(\sigma_v''). \tag{3.3.20}$$

Note in particular that we have

$$\tilde{\omega}_{12} - (-1)^n \tilde{\omega}_{21} + \sum \pm \varepsilon_\partial(\sigma'_1 \sigma'_2)\sigma''_1 \otimes \sigma''_2 = 0 \in \tilde{\mathtt{G}}_A(\underline{2}), \tag{3.3.21}$$

with the last summand being a cycle. The relations for $\#T = 2$ are perturbations of the symmetry relation of $\mathtt{G}_A(V)$, and the relations for $\#T = 3$ are perturbations of the Arnold relations in $\mathtt{e}_n^\vee(V)$ (where, by "perturbation", we informally mean that the perturbed relation is the sum of the standard relation and terms which have a strictly lower number of generators $\tilde{\omega}_{vv'}$).

Example 3.3.22. Let us consider $M = S^1 \times [0, 1]$ (even though it doesn't satisfy our assumptions about connectivity). We can find a PLD model for M where:

- $B_\partial = H^*(\partial M)$ is four-dimensional, generated by 1, t, $d\varphi$, and $td\varphi$ with t being idempotent;
- $A = H^*(M) = H^*(S^1)$ is two-dimensional, generated by 1 and $d\varphi$;
- $K = H^*(M, \partial M)$ is two-dimensional, generated by dt and $dt \wedge d\varphi$.

Then one of the nontrivial relation in $\tilde{\mathtt{G}}_A(2)$ is given by $(d\varphi \otimes 1)\tilde{\omega}_{21} + (1 \otimes d\varphi)\tilde{\omega}_{12} + (d\varphi \otimes d\varphi) = 0$. To obtain some intuition about this relation, consider that $\mathrm{Conf}_2(M) \simeq \mathrm{Conf}_2(\mathbb{R}^2 - \{0\})$ is homotopy equivalent to $\mathrm{Conf}_3(\mathbb{R}^2)$. The element $d\varphi$ correspond to the two points are rotating around the origin. Thus we can identify $\tilde{\omega}_{12}$ with $\omega_{12} \in H^*(\mathrm{Conf}_3(\mathbb{R}^2))$, $d\varphi \otimes 1$ with ω_{13}, and $1 \otimes d\varphi$ with ω_{23}. The perturbed relation in $\tilde{\mathtt{G}}_A(2)$ is then nothing but the usual Arnold relation in $\mathtt{e}_2^\vee(3) = H^*(\mathrm{Conf}_3(\mathbb{R}^2))$.

Proposition 3.3.23. *There is an isomorphism of dg-modules between* $\mathtt{G}_A(V)$ *and* $\tilde{\mathtt{G}}_A(V)$.

Proof. Let us fix $V = \{1, \dots, k\}$ for some $k \geq 0$. Consider the standard basis of $\mathtt{e}_n^\vee(k)$ given by monomials of the type:

$$\omega_{i_1 j_1} \dots \omega_{i_r j_r},$$

with $1 \leq i_1 < \dots < i_r \leq r$ and $i_l < j_l$ for all l. By choosing some basis $\{a_1, \dots, a_m\}$ of A, we obtain a basis of $\mathtt{G}_A(k)$ by labeling the last element of each connected component of such a monomial by some a_i.

We claim that if we replace all the ω_{ij} by $\tilde{\omega}_{ij}$ in this basis, then we obtain a basis of $\tilde{\mathtt{G}}_A(V)$. Using the perturbed Arnold relations, it's clear that any element of $\tilde{\mathtt{G}}_A(V)$ can be written as a linear combination of these elements. Moreover, using the same argument that proves that there is no nontrivial relation between the elements of the standard basis of $\mathtt{e}_n^\vee(k)$, we can prove that there is non nontrivial relation between the elements of our claimed basis.

There is thus a linear isomorphism $\mathtt{G}_A(V) \to \tilde{\mathtt{G}}_A(V)$ which is defined on the basis by replacing all the ω_{ij} by $\tilde{\omega}_{ij}$. It's then clear that this map preserves the internal differential of A and the part of the differential which splits an $\omega_{i_l j_l}$ (which can be written down explicitly in the basis: it merely splits a connected component into two). □

Corollary 3.3.24. *There is an isomorphism of graded vector spaces* $H^*(\mathrm{Conf}_k(M)) \cong H^*(\tilde{\mathtt{G}}_A(k))$.

Proof. This follows immediately from Theorem 3.3.16 □

Remark 3.3.25. It is often the case that $\mathtt{G}_A$ and $\tilde{\mathtt{G}}_A$ are actually equal. For example, if M is obtained by removing a point from a closed manifold, then $B_\partial = H^*(S^{n-1}) = S(v)/(v^2)$, and $\sigma_A = v \otimes 1$. Then in all the "corrective terms" in the definition of $\tilde{\mathtt{G}}_A(V)$, the power of v is at least two, hence the term vanishes.

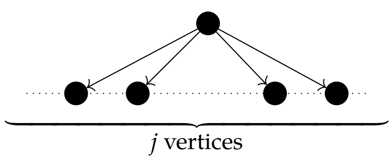
Example 3.3.26. If $M = D^n$ and we use the PLD model of Example 3.1.16, then $\tilde{\mathtt{G}}_A = \mathtt{G}_A$ (see Example 3.3.5).

3.4 Computation of Swiss-Cheese graph cohomology

In this section, we prove statements related to the vanishing of the homology of a twist of $\mathsf{SGC}_n^\vee$ (see Section 3.1.5). We assume that $n \geq 3$ throughout the section.

Elements of SGC_n are given by linear combinations of graphs; we identify an element in the dual with a (possibly infinite) sum of graphs using the dual basis. Let δ be the differential in $\mathsf{SGC}_n^\vee$: it is given by "splitting" an aerial vertex (in all possible ways) in two vertices connected by an edge. Then $\mathsf{SGC}_n^\vee$ is a (complete) dg-Lie algebra, with the bracket given by $[\gamma, \gamma'] = \gamma \circ \gamma'' - \pm\gamma' \circ \gamma$, where $\gamma \circ \gamma'$ is defined by the sum over the terrestrial vertices of γ of the insertion of γ' into that terrestrial vertex and reconnecting the incident edges in all possible ways.

Recall the Kontsevich integrals $c \in \mathsf{SGC}_n^\vee$ (see Equation (3.1.24)). It is well-known (see e.g. [Wil15, Section 6.1]) that c starts with a linear part c_0 corresponding to the Hochschild–Kostant–Rosenberg map, which we now describe. Let γ_j, for $j \geq 0$, be the graph with one aerial vertex, j terrestrial vertex, and an edge from the aerial vertex to each terrestrial vertex.

$\gamma_j :=$  (3.4.1)

Definition 3.4.2. The linear part $c_0 \in \mathrm{SGC}_n^\vee$ of c is given by $c_0(\gamma_j) = \frac{1}{j!}$ and $c_0(\gamma) = 0$ if γ does not have exactly one aerial vertex.

The element c_0 itself is a Maurer–Cartan element in $SGC_n^\vee$. In other words, $c - c_0$, which vanishes on all graphs with exactly one aerial vertex, is a Maurer–Cartan in the twisted Lie algebra

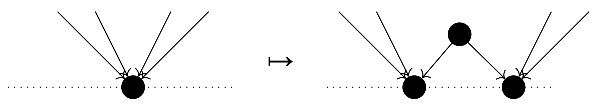
$$\mathrm{SGC}_n^{\vee, c_0} := (\mathrm{SGC}_n^\vee, \delta + [c_0, -]). \tag{3.4.3}$$

We'd like to prove that c is gauge equivalent to c_0, or equivalently that $c - c_0$ is gauge trivial in the twisted Lie algebra.

Proposition 3.4.4. *For each $i \in \mathbb{Z}$, the dimension of $H^i(\mathrm{SGC}_n^{\vee, c_0})$ is at most equal to the dimension of $H^i(\mathrm{GC}_n^\vee) \oplus H^i(\mathrm{GC}_{n-1}^\vee)$.*

This is to be compared with a similar result for $n = 2$ at the beginning of [Wil15, Section 5], stating that $H^*(\mathrm{SGC}_2^{\vee, c_0}) \cong H^*(\mathrm{GC}_2^\vee)$. This result relies in turn on [Wil16, Appendix F], and we will use similar proof techniques.

Proof of Proposition 3.4.4. Call an aerial vertex a "wedge" if it is bivalent and connected to two terrestrial vertices. For example, the only aerial vertex of γ_2 is a wedge. We define a complete decreasing filtration on $\mathrm{SGC}_n^{\vee, c_0}$ by the number of aerial vertices that are *not* wedges. On the E^0 page, the differential is given by the bracket $\gamma \mapsto [\gamma_2, -]$, which can be represented as follows:



This roughly corresponds to the differential in $\mathtt{Graphs}_{n-1}^\vee$. Note that the creations of "dead ends" in $\gamma \circ \gamma_2$ (i.e. when all the incoming edges at a vertex are put on one side of the wedge) are canceled with the summands coming from $\gamma_2 \circ \gamma$.

Our goal is to prove that on $\mathsf{E}^1 = H(\mathsf{E}^0, d^0)$, there only remains univalent terrestrial vertices ("hairs") and no wedges. The computation is conceptually similar to the one right after [Wil16, Claim 1], except that instead of computing the Hochschild homology of a symmetric algebra, we compute the e_{n-1}-cohomology of a symmetric algebra with trivial bracket.

The E^0 page splits in a direct sum as follows. Given some $\gamma \in \mathsf{E}^0$, we consider the "index" $\mathrm{idx}(\gamma)$ obtained by removing all wedges and cutting off the terrestrial vertices, keeping a half-edge. We obtain in this way a possibly disconnected graph with only aerial vertices, each one having a certain number of half-edges attached to it. The only restriction is that a zero-valent vertex may not have two half-edges attached to it (otherwise it would be a wedge). Then $\mathsf{E}^0_\Gamma := \{\gamma \in \mathsf{E}^0 \mid \mathrm{idx}(\gamma) = \Gamma\}$ is a subcomplex, and that $\mathsf{E}^0 = \bigoplus_\Gamma \mathsf{E}^0_\Gamma$.

We'd like to compute $H^*(\mathsf{E}^0_\Gamma, d^0)$. There are two cases to consider. If $\Gamma = \emptyset$, then $(\mathsf{E}^0_\emptyset, d^0)$ is isomorphic to $\mathrm{GC}^\vee_{n-1}$ in the following way. If $\gamma \in \mathrm{GC}^\vee_{n-1}$, then we can view it as an element of $\mathsf{E}^0_\emptyset$ by viewing all its vertices as terrestrial, and by replacing all its edges by wedges. The differential d^0 is then the differential of $\mathrm{GC}^\vee_{n-1}$.

Now if $\Gamma \neq \emptyset$, then the actual shape of Γ doesn't matter, only the number of half-edges does. If we number the half-edges as $h_1, \dots, h_k$, then we see that we are actually computing the $\mathtt{Graphs}^\vee_{n-1}$-cohomology of the symmetric algebra $S(h_1, \dots, h_k)$ and taking the summand where each half-edge appears exactly once. Since $\mathtt{Graphs}^\vee_{n-1} \simeq \mathsf{e}_{n-1}$, we have $H^*(\mathsf{E}^0_\Gamma) = H^*_{\mathsf{e}_{n-1}}(S(h_1, \dots, h_k); \mathbb{R})^{(1,\dots,1)}$. Applying the higher HKR theorem, this cohomology is given by a symmetric algebra on a shift of the generators. Going back to E^0_Γ, under this identification, $H^*(\mathsf{E}^0_\Gamma, d^0)$ is given by graphs where the code is Γ,there are no wedges, and each terrestrial vertex is univalent.

In other words, we can identify E^1 as a direct sum of $H^*(\mathrm{GC}^\vee_{n-1})$ and a version of the hairy graph complex (as a vector space) where each vertex may have multiple hairs. The differential d^1 is given by the part of the differential of $\mathrm{SGC}^{\vee,c_0}_n$ which raises the number of non-wedge vertices by exactly one, i.e. $d^1 = \delta + [\gamma_0, -] + [\gamma_1, -] + \sum_{j\geq 3}[\gamma_j, -]$.

We can now apply the same technique which appears in the proof of [Wil16, Claim 2]. The differential d^1 splits into three parts: $d^{1,-1} = [\gamma_0, -]$ which decreases the number of terrestrial vertices by 1, $d^{1,0} = \delta + [\gamma_1, -]$ which keeps it constant, and $d^{1,\geq 1} = \sum_{j\geq 3}[\gamma_j, -]$ which increases it. Let $\mathsf{E}^{1'}$ be the quotient of E^1 by graphs with terrestrial vertices as well as graphs in the image of $d^{1,-1}$ (i.e. graphs with univalent aerial vertices connected to another aerial vertex). The map $d^{1,0}$ induces a differential on $\mathsf{E}^{1'}$. We can then reuse the proof of [Wil16, Claim 2] to prove that (E^1, d^1) is quasi-isomorphism to the direct sum of $H^*(\mathrm{GC}^\vee_{n-1})$ and the quotient $(\mathsf{E}^{1'}, d^{1,0})$.

Then $(\mathsf{E}^{1'}, d^{1,0} = \delta)$ has the same cohomology as $\mathrm{GC}^\vee_n$, using an argument identical to the one in [Wil16, Appendix F.2.1]. Thus, the E^2 page is a direct sum of $H^*(\mathrm{GC}^\vee_n)$ and $H^*(\mathrm{GC}^\vee_{n-1})$. It follows that the dimension of the cohomology of $\mathrm{SGC}^{\vee,c_0}_n$ is at most the dimension of the cohomology of $\mathrm{GC}^\vee_n \oplus \mathrm{GC}^\vee_{n-1}$. □

Recall that $H^*(\mathrm{GC}^\vee_n)$ and $H^*(\mathrm{GC}^\vee_{n-1})$ contain some special classes, the circular graphs (or loops, but this terminology clashes with our terminology for edges between a vertex and itself), see e.g. [Wil14, Proposition 3.4]. We depict these circular graphs in Figure 3.4.1.

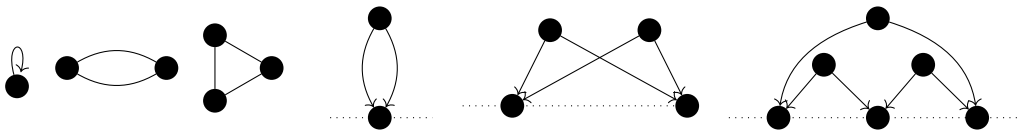


Figure 3.4.1: Circular graphs in GC_n and GC_{n-1}

Corollary 3.4.5. *The Maurer–Cartan element c is gauge equivalent to c_0 in $\mathrm{SGC}_n^\vee$.*

Proof. According to [FW15, Proposition 2.2.3], the cohomology of $\mathrm{GC}_n^\vee$ is given by a sum of the circular graphs and a part which vanishes in degrees $* > -n$ (we work with cohomological conventions, so the degree is reversed). The integral defining c vanishes on all the circular graphs for degree reasons: the total degree of a circular graph with k "edges" (either an edge between two aerial vertices or a wedge between two terrestrial vertices) is $-k < 0$. It then follows from Proposition 3.4.4 that $c - c_0$ lives in a Lie subalgebra of $\mathrm{SGC}_n^{\vee,c_0}$ whose cohomology vanishes in degrees $* > -(n-1)$. Hence, by obstruction theory,[4] the Maurer–Cartan element $c - c_0$ is gauge trivial. $\square$

The definition of the cooperad $\mathtt{SGraphs}_n$ (cf. Section 3.1.4) depends on the choice of this Maurer–Cartan element, c or c_0.

Definition 3.4.6. The cooperad $\mathtt{SGraphs}_n^{c_0}$ is defined similarly to $\mathtt{SGraphs}_n$, but the Maurer–Cartan element c_0 is used in place of c in the definition of the differential.

By choosing a path object for $\mathrm{SGC}_n^\vee$, we can turn the gauge equivalence between c and c_0 into:

Corollary 3.4.7. *There is a zigzag of quasi-isomorphism of cooperads:*

$$\mathtt{SGraphs}_n \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathtt{SGraphs}_n^{c_0}. \quad \square$$

Let us also record the following observation. The comodule $\mathtt{SGraphs}_n^{c_0}(\emptyset, -)$ is not isomorphic to $\mathtt{Graphs}_n$ seen as a comodule over itself, because its graphs can contain internal terrestrial vertices. However, $\mathtt{SFM}_n(\emptyset, \{*\})$ is a point, and there is a weak equivalence of $\mathtt{FM}_n$-modules given by:

$$\mathtt{FM}_n(U) \cong \mathtt{SFM}_n(\emptyset, \{*\}) \times \mathtt{FM}_n(U) \xrightarrow{\circ_U} \mathtt{SFM}_n(\emptyset, U). \tag{3.4.8}$$

This is modeled by the following proposition. Define the map

$$\nu : \mathtt{SGraphs}_n^{c_0}(\emptyset, V) \to \mathtt{Graphs}_n(V) \tag{3.4.9}$$

[4] The $\mathtt{E}^1$ page of the spectral sequence computing $\pi_* \operatorname{Map}_{\mathrm{CDGA}}((S(\mathrm{SGC}_n[1]), \delta_{c_0}), \mathbb{R}) = \pi_* \mathrm{MC}_\bullet(\mathrm{SGC}_n^{\vee,c_0})$ vanishes in the right degrees.

as follows. Given some $\Gamma \in \mathsf{SGraphs}_n^{c_0}(\emptyset, V)$, if Γ only has univalent terrestrial vertices, then $\nu(\Gamma)$ is the graph with these univalent vertices and their incident edges removed. Otherwise, if Γ has terrestrial vertices of valence greater than one, then $\nu(\Gamma) = 0$.

Proposition 3.4.10. *The maps $\nu : \mathsf{SGraphs}_n^{c_0}(\emptyset, V) \to \mathsf{Graphs}_n(V)$ defines a quasi-isomorphism of Hopf right Graphs_n-modules.*

Proof. For a given V, the map is the composite of two quasi-isomorphisms of CDGAs (hence it is itself a CDGA map and a quasi-isomorphism):

- The comodule structure map

$$\circ_V^\vee : \mathsf{SGraphs}_n^{c_0}(\emptyset, V) \to \mathsf{SGraphs}_n^{c_0}(\emptyset, \{*\}) \otimes \mathsf{Graphs}_n(V),$$

  which is a quasi-isomorphism because it models the weak equivalence $\mathsf{FM}_n(\emptyset, \{*\}) \times \mathsf{FM}_n(V) \to \mathsf{SFM}_n(\emptyset, V)$;

- The map $\mathsf{SGraphs}_n^{c_0}(\emptyset, \{*\}) \to \mathbb{R}$ given by c_0, tensored with the identity of $\mathsf{Graphs}_n(V)$. This is a quasi-isomorphism because $\mathsf{SGraphs}_n^{c_0}(\emptyset, \{*\})$ is a model for $\mathsf{SFM}_n(\emptyset, \{*\})$, which is a point, and c_0 is a nontrivial cocycle.

The fact that $\mathsf{SGraphs}_n^{c_0}$ is a relative operad over Graphs_n also shows that this is a morphism of Graphs_n-comodules. □

3.5 A model of right Hopf comodules

3.5.1 The propagator

The proof of Theorem C in the case of closed manifolds relied on the existence of a propagator, see Section 2.3.3. Let us briefly recall its construction in this case, before going back to the case $\partial M \neq \emptyset$. The propagator φ is an $(n-1)$-form on $\mathsf{FM}_M(\underline{2})$ whose differential was the pullback of the diagonal class. Moreover, recall that the projection $\partial \mathsf{FM}_M(\underline{2}) \to M$ is an S^{n-1}-bundle, and the restriction of the propagator to the boundary is a global angular form for this bundle. If M is framed then this bundle is trivial (with $\circ_1 : M \times S^{n-1} \to \mathsf{FM}_M(\underline{2})$ being an isomorphism of bundles), and one can further assume that φ is equal to $1 \times \mathrm{vol}_{n-1}$.

This propagator was constructed by considering a global angular form of the previous bundle, pulling it back to a tubular neighborhood of the boundary inside $\mathsf{FM}_n(\underline{2})$, multiplying it by a bump function, and then extending it by zero outside of the tubular neighborhood (see [CW16, Proposition 7]). Moreover, it can be chosen so that it belongs to the subalgebra $\Omega^*_{\mathrm{triv}}(\mathsf{FM}_M(\underline{2})) \subset \Omega^*_{\mathrm{PA}}(\mathsf{FM}_M(\underline{2}))$

of "trivial" PA forms (see [CW16, Appendix C]), which implies that it can be integrated along the fibers of the canonical projections p_V.

Let us now return to the case $\partial M \neq \emptyset$. Consider the double $D = M \cup_{\partial M} \bar{M}$, a closed manifold. One can construct a propagator $\psi \in \Omega_{\mathrm{triv}}(\mathsf{FM}_D(\underline{2}))$. There is an "inclusion" map $i : \mathsf{SFM}_M(\emptyset, \underline{2}) \to \mathsf{FM}_D(\underline{2})$ (which is not actually injective on the set of configurations $x \in \mathsf{SFM}_M(\underline{2})$ where $p_1(x) = p_2(x) \in \partial M$, just like the standard map $\mathsf{SFM}_n(\emptyset, \underline{2}) \to \mathsf{FM}_n(\underline{2})$ is not injective). Moreover, there is a "mirror" map $\tau_1 : \mathsf{SFM}_M(\emptyset, \underline{2}) \to \mathsf{FM}_D(\underline{2})$ which is defined similarly, except that the first point of the configuration is sent to its mirror in $\bar{M} \subset D$. We then define the propagator on M to be:

$$\varphi := \frac{1}{2}(i^* \psi - \tau_1^* \psi). \tag{3.5.1}$$

One can write down the cohomology of $H^*(D)$ in terms of the cohomology of M and ∂M and check on a basis that $[d\varphi] \in H^*(M \times M)$ is indeed the diagonal class of M. Note that this propagator is not symmetrical: if the point 1 (the "origin" of the propagator) becomes infinitesimally close to ∂M, then the two contributions cancel out and φ become zero; this is not the case for the point 2 (the "target").

Consider the space:

$$E = \{x \in \mathsf{SFM}_M(\emptyset, \underline{2}) \mid p_1(x) = p_2(x)\} \tag{3.5.2}$$

which is an S^{n-1}-bundle $\mathring{E}$ when restricted over the interior of M. Given that ψ is a global angular form on $\partial \mathsf{FM}_D(\underline{2})$, we check that φ is a global angular form on this S^{n-1}-bundle.

Remark 3.5.3. We go back to the construction of the diagram of Equation (3.2.17) and replace $\Omega^*_{\mathrm{PA}}(-)$ with its quasi-isomorphic subalgebra $\Omega^*_{\mathrm{triv}}(-)$, and then build a PLD model from that. We can then compose with the inclusion $\Omega^*_{\mathrm{triv}}(-) \to \Omega^*_{\mathrm{PA}}(-)$ and we obtain a diagram:

$$\begin{tikzcd}
R \arrow[r, "\sim", "f"'] \arrow[d, "\rho"] & \Omega^*_{\mathrm{triv}}(M) \arrow[r, hook, "\sim"] \arrow[d, "\mathrm{res}"] & \Omega^*_{\mathrm{PA}}(M) \arrow[d, "\mathrm{res}"] \\
R_\partial \arrow[r, "\sim", "f_\partial"'] & \Omega^*_{\mathrm{triv}}(M) \arrow[r, hook, "\sim"] & \Omega^*_{\mathrm{PA}}(M)
\end{tikzcd}$$

This allows to assume that if $x \in R$ (resp. $y \in R_\partial$), then $g(x)$ (resp. $g_\partial(y)$) is a trivial form on M (resp. ∂M) – compare with Remark 2.3.18.

Let us fix the differential of the resulting propagator. Recall the diagonal class $\Delta_{AK} \in A \otimes K$ from Equation (3.2.21), which gets sent to $\Delta_A \in A \otimes A$ under $K \subset B \xrightarrow{\pi} A$. Then we have:

Lemma 3.5.4. *There exists an element $\Delta_R \in R \otimes R$ such that $(\pi \otimes \mathrm{id})g^{\otimes 2}(\Delta_R) = \Delta_{AK}$ and $\mu_R(\Delta_R) = 0$.*

Proof. The proof is identical to the proof of Proposition 2.3.4, recalling that $R \to B$ and $B \to A$ are both surjective, and that $\mu_A(\Delta_A) = 0$ when $\partial M \neq 0$ (Equation (3.2.23)). □

Definition 3.5.5. For convenience, define a cocycle Δ_B in $B^{\otimes 2}$ by:

$$\Delta_B := g^{\otimes 2}(\Delta_R).$$

Finally, we'd like to fix the value of the propagator on the subspace $\mathsf{SFM}_M(\{2\},\{1\})$ of $\mathsf{SFM}_M(\emptyset,\{1,2\})$. In order to define the inclusion

$$j : \mathsf{SFM}_M(\{2\},\{1\}) \hookrightarrow \mathsf{SFM}_M(\emptyset,\{1,2\}), \tag{3.5.6}$$

we choose some inward pointing vector field on ∂M and we use it to infinitesimally push the second point into the interior of M.

Recall the element $\sigma_B \in B_\partial \otimes B$ defined in Section 3.3.3, with $d\sigma_B \in B_\partial \otimes K$ being cohomologous to the diagonal class of M with the first factor restricted to ∂M. Using the fact that we have a surjective quasi-isomorphism (thanks to the five lemma):

$$\mathrm{cone}(R_\partial \otimes \ker \rho) \to \mathrm{cone}(B \otimes K), \tag{3.5.7}$$

we can find some element $\sigma_R \in R_\partial \otimes R$ with $(g_\partial \otimes g)(\sigma_R) = \sigma_B$ and $d\sigma_R \in R_\partial \otimes \ker \rho$. Then $j^*\varphi - (f_\partial \otimes f)(\sigma_R)$ is an exact form, by checking on cohomology.

Proposition 3.5.8. *There exists a form $\varphi \in \Omega^{n-1}_{\mathrm{PA}}(\mathsf{SFM}_M(\emptyset,\underline{2}))$ such that:*

- *$d\varphi$ is the pullback of $(f \otimes f)(\Delta_R)$ along the product of the projections $\mathsf{SFM}_M(\underline{2}) \to M \times M$;*
- *its restriction to the bundle $\mathring{E}$ from Equation (3.5.2) is a global angular form;*
- *if moreover M is framed we choose φ so that $\varphi|_{\mathring{E}} = 1 \times \mathrm{vol}_{n-1}$;*
- *$j^*\varphi = \varphi|_{\mathsf{SFM}_M(\{2\},\{1\})} = (f_\partial \otimes f)(\sigma_R)$;*
- *for any $x \in R$, one has $(p_2)_*(p_1^*(x)\varphi) = 0$.*

Proof. We've already seen how to construct a propagator satisfying the first three properties, using the mirror $D = M \cup_{\partial M} \bar{M}$.

The difference $j^*\varphi - (f_\partial \otimes f)(\sigma_R)$ is an exact form, say $d\xi$ where ξ is a trivial form on $\mathsf{SFM}_M(\{1\},\{2\})$. The application j^* is surjective (because $\mathsf{SFM}_M(\{1\},\{2\})$ is a submanifold with corners of $\mathsf{SFM}_M(\emptyset,\{1,2\})$), hence there exists some $\hat{\xi}$ such that $j^*\hat{\xi} = \xi$. It then suffices to replace φ by $\varphi - d\hat{\xi}$, which still satisfies the first three properties.

Finally, for the last property, we reuse the proof technique of [CM10, Lemma 3]. Namely, we consider the chain map:

$$\begin{aligned} r : \Omega^*_{\mathrm{triv}}(M) &\to R \\ x &\mapsto \sum_{(\Delta_R)} \left(\int_M x\Delta'\right)\Delta'' + \sum_{(\sigma_R)} \left(\int_{\partial M} x|_{\partial M}\sigma'\right)\sigma'', \end{aligned}$$

satisfying $f = f \circ r \circ f$, and the homotopy:

$$\begin{aligned} h : \Omega^*_{\mathrm{triv}}(M) &\to \Omega^{*-1}_{\mathrm{triv}}(M) \\ x &\mapsto (p_1)_*(p_2^*(x) \wedge \varphi). \end{aligned}$$

Then we replace h by $h' := (\mathrm{id} - fr) \circ h \circ (\mathrm{id} - fr)$ in order to obtain a new homotopy satisfying $h'''' \circ f = 0$. Let us write down the integral kernel of h'. In order to simplify notations, we write $\int_{i,j,\ldots}(-)$ to indicate that we push forward along the projection which forgets the points $i, j, \ldots$, we write $\Delta_{ij} = p_{ij}^*(\Delta_R)$ and $\sigma_{ij} = p_{ij}^*(\sigma_R)$. Then the new propagator is:

$$\begin{aligned} \tilde{\varphi}_{12} := {} & \varphi_{12} - \int_3 \varphi_{23}\Delta_{13} - \int_3 \varphi_{23}\sigma_{13} - \int_3 \varphi_{13}\Delta_{23} - \int_3 \varphi_{13}\sigma_{23} \\ & + \int_{3,4} \Delta_{24}\varphi_{34}\Delta_{13} + \int_{3,4} \sigma_{24}\varphi_{23}\Delta_{13} + \int_{3,4} \Delta_{24}\varphi_{34}\sigma_{13} + \int_{3,4} \sigma_{42}\varphi_{34}\sigma_{13}. \ \square \end{aligned}$$

By construction, this new propagator $\tilde{\varphi}$ satisfies the last property. To check that the corrective term $\tilde{\varphi} - \varphi$ is a cycle, one needs to apply the Stokes formula and check that all the terms cancel out, which follows from the general properties of the diagonal class, the fact that $\varphi|_{\hat{E}}$ is a global angular form, and that $j^*\varphi = \sigma$. Similarly to check that $j^*(\tilde{\varphi} - \varphi) = 0$ and that $(\tilde{\varphi} - \varphi)|_{\hat{E}}$, one need to use the functoriality property of integral along fibers [Har+11, Proposition 8.9] and check that all the terms cancel out.

3.5.2 Colored labeled graphs

We now introduce a colored version of the graph comodule $\mathtt{Graphs}_R$ considered in Section 2.3. Recall the cocycle Δ_R from Lemma 3.5.4. We also define $\Delta_{R,R_\partial} = (\mathrm{id} \otimes \rho)(\Delta_R) \in R \otimes R_\partial$ and $\Delta_{R,\Omega^*(\partial M)} := (\mathrm{id} \otimes g_\partial)(\Delta_{R,R_\partial})$ for notational convenience.

Definition 3.5.9. The CDGA of $(R, \Omega^*_{\mathrm{triv}}(\partial M))$-labeled graphs on the finite sets U, V is:

$$\mathtt{SGra}_{R,\Omega^*(\partial M)}(U, V) := ((\Omega^*_{\mathrm{triv}}(\partial M))^{\otimes U} \otimes R^{\otimes V} \otimes \mathtt{SGra}_n(U, V), d)$$

with differential defined by $de_{vv''''} = \iota_{vv'}(\Delta_R)$ for $v, v' \in V$, and $de_{vu} = \iota_{vu}(\Delta_{R,\Omega^*(\partial M)})$ for $u \in U$ and $v \in V$.

Proposition 3.5.10. *The bisymmetric collection* $\mathtt{SGra}_{R,\Omega^*(\partial M)}$ *is a Hopf right comodule over* $\mathtt{SGra}_n$*.*

Proof. The proof is identical to the proof of Proposition 2.3.7. Recall that we do not need additional assumptions on the Euler characteristic as soon as we assume $\partial M \neq \emptyset$ (see Lemma 3.5.4). □

This comodule has a graphical description similar to $\mathtt{SGra}_n$. The difference is that aerial points (corresponding to the interior of M) are labeled by R, while terrestrial points (corresponding to the boundary) are labeled by R_∂.

Given a graph Γ, the differential $d\Gamma$ is a sum over the set of edges of Γ. For each summand, one removes the edge from the graph and multiplies the endpoints of the edge by either Δ_R (if both endpoints are aerial) or $\Delta_{R,\partial}$ (if one endpoint is aerial and the other terrestrial). We call this "splitting" the edge, see Figure 3.5.1. Recall that we write $\Delta_R = \sum_{(\Delta_R)} \Delta'_R \otimes \Delta''_R \in R^{\otimes 2}$.

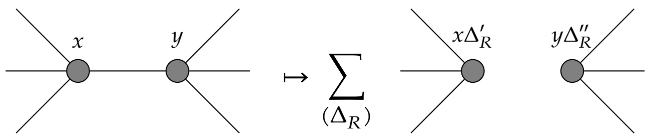


Figure 3.5.1: The splitting differential of $\mathtt{SGra}_{R,\Omega^*(\partial M)}$ (a gray vertex can be of any kind).

The product glues graphs along their vertices (multiplying the labels), and the comodule structure collapses subgraphs, multiplying the labels and applying ρ to them if necessary.

Proposition 3.5.11. *There is a morphism of bisymmetric collections* $\omega' : \mathtt{SGra}_{R,\Omega^*(\partial M)} \to \Omega^*_{\mathrm{PA}}(\mathtt{SFM}_M)$ *characterized by*

$$
\begin{aligned}
\omega'(\iota_v(x)) &= p_v^*(g(x)) & v \in V,\ x \in R \\
\omega'(\iota_u(x)) &= p_u^*(x) & u \in U,\ x \in \Omega^*_{\mathrm{triv}}(\partial M) \\
\omega'(e_{vv'}) &= p_{vv'}^*(\varphi) & v, v' \in V \\
\omega'(e_{vu}) &= p_{vu}^*(j^*(\varphi)) & u \in U,\ v \in V
\end{aligned}
$$

where $j : \mathsf{SFM}_M(\underline{1}, \underline{1}) \to \mathsf{SFM}_M(\emptyset, \underline{2})$ is the inclusion.[5] *Moreover if M is framed, then together with Willwacher's morphism [Wil15], this defines a Hopf right comodule morphism*

$$\omega' : (\mathsf{SGra}_{R,\Omega^*(\partial M)}, \mathsf{SGra}_n) \to (\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M), \Omega^*_{\mathrm{PA}}(\mathsf{SFM}_n)).$$

Proof. The proof is identical to the proof of Proposition 2.3.11. □

Let us define a Maurer–Cartan element $c_\varphi \in \mathrm{SGC}_n^\vee \otimes \Omega^*_{\mathrm{triv}}(\partial M)$. Let $\gamma \in \mathrm{SGC}_n$ be a connected graph, let I be the set of its terrestrial vertices and J of its aerial vertices. Then γ defines a form $\omega'(\gamma)$ on $\mathsf{SFM}_M(I, J)$, which we can pull back along the composition map $\circ_{I,J} : \partial M \times \mathsf{SFM}_n(I, J) \to \mathsf{SFM}_M(I, J)$ and then integrate on the $\mathsf{SFM}_n(I, J)$ factor, in other words:

$$c_\varphi(\gamma) := (p_{\partial M})_*(\circ_{I,J}^*(\omega'(\gamma))). \tag{3.5.12}$$

Remark 3.5.13. For all γ, $\omega'(\gamma)$ is a trivial form (in the sense of [CW16, Appendix C]), hence $c_\varphi(\gamma)$ is obtained by pushing forward a trivial form along a trivial bundle and is therefore trivial itself.

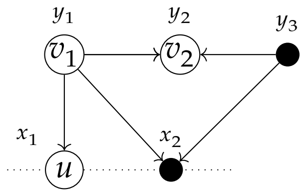
Definition 3.5.14. The **twisted colored labeled graph comodule** $\mathrm{Tw}\,\mathsf{SGra}_{R,\Omega^*(\partial M)}$ is the Hopf right $(\mathrm{Tw}\,\mathsf{SGra}_n)$-comodule obtained by twisting $\mathsf{SGra}_{R,\Omega^*(\partial M)}$ with respect to the Maurer–Cartan element c_φ.

Let us give a graphical description of the module $\mathrm{Tw}\,\mathsf{SGra}_{R,\Omega^*(\partial M)}(U, V)$. It is spanned by graphs with four types of vertices: on the one hand, either aerial or terrestrial, and on the other hand either external or internal. Recall that aerial vertices are labeled by R while terrestrial ones are labeled by R_∂. External aerial vertices are in bijection with V, while external terrestrial vertices are in bijection with U. Internal aerial vertices are of degree $-n$, while internal terrestrial vertices are of degree $-(n-1)$. Both kinds of internal vertices are indistinguishable among themselves. Finally, edges are of degree $n-1$, and the source of an edge may only be aerial.

The product glues graphs along external vertices (multiplying the labels). The comodule structure maps collapse subgraphs, and the label of the collapsed subgraph is the product of all the labels inside that subgraph (applying $\rho : R \to R_\partial$ as needed). Finally, the differential has several parts:

- A first part comes from $\mathsf{SGra}_{R,\Omega^*(\partial M)}$: it splits edges between vertices of any type, and then multiplies the endpoints of the removed edge by either Δ_R or $\Delta_{R,\partial}$.

[5] Recall that one must choose a collar $\partial M \times [0, 1) \subset M$ in order to define this inclusion.



$(x_1, x_2 \in R_\partial$ and $y_1, y_2, y_3 \in R)$

Figure 3.5.2: A colored labeled graph in Tw $\mathtt{SGra}_{R,\Omega^*(\partial M)}(\{u\}, \{v_1, v_2\})$.

- A second part contracts edges between two aerial vertices, one of them being internal, and multiplies the labels of the endpoints in the process. This includes dead ends, i.e. edges connected to a univalent internal vertex.

- A third part contracts a subgraph Γ' with at most one external vertex. The label of the contracted subgraph is the product of all the labels inside it multiplied by $c_\varphi(\bar{\Gamma}')$, where $\bar{\Gamma}'$ is the subgraph with the labels removed. If the result contains a "bad edge" (whose source is terrestrial), then the summand vanishes.

Remark 3.5.15. Note that there are several differences compared to the description of the differential from Section 3.1.4: dead ends are contractible, and the last part of the differential which forgets some internal vertices is not present. This comes from the fact that in the definition of the twisting of a right comodule over a cooperad, the Maurer–Cartan element of the deformation complex can only act "from the right" on the comodule.

Dead ends aren't contractible in Tw $\mathtt{SGra}_n$ because the contraction of a dead end appears twice in the differential, one "from the left" and one "from the right", and they cancel each other (see [Wil14, Appendix I.3]). This cancellation does not occur in Tw $\mathtt{SGra}_{R,\Omega^*(\partial M)}$. The last part of the differential on Tw $\mathtt{SGra}_n$ came exclusively from the left action of the deformation complex on $\mathtt{SGra}_n$, and so cannot appear in Tw $\mathtt{SGra}_{R,\Omega^*(\partial M)}$.

Remark 3.5.16. One should not forget that when a subgraph $\Gamma' \subset \Gamma$ with only internal vertices is contracted (the third part), the result may be a terrestrial vertex even though the subgraph contains only aerial vertices, see Figure 3.5.3.

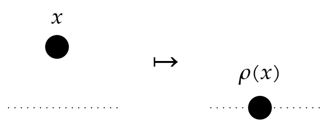


Figure 3.5.3: Collapsing an aerial vertex into a terrestrial vertex

Proposition 3.5.17. *The morphism $\omega' : \mathsf{SGra}_{R,\Omega^*(\partial M)} \to \Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M)$ of Proposition 3.5.11 extends to a morphism of bisymmetric collections $\omega : \mathrm{Tw}\,\mathsf{SGra}_{R,\Omega^*(\partial M)} \to \Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M)$, given on a graph $\Gamma \in \mathsf{SGra}_{R,\Omega^*(\partial M)}(U \sqcup I, V \sqcup J) \subset \mathrm{Tw}\,\mathsf{SGra}_{R,\Omega^*(\partial M)}(U,V)$ by:*

$$\omega(\Gamma) := (p_{U,V})_*(\omega'(\Gamma)) = \int_{\mathsf{SFM}_M(U \sqcup I, V \sqcup J) \to \mathsf{SFM}_M(U,V)} \omega'(\Gamma).$$

Moreover if M is framed then, together with Willwacher's morphism [Wil15], this assignment defines a morphism of Hopf right comodules

$$\omega : (\mathrm{Tw}\,\mathsf{SGra}_{R,\Omega^*(\partial M)}, \mathrm{Tw}\,\mathsf{SGra}_n) \to (\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M), \Omega^*_{\mathrm{PA}}(\mathsf{SFM}_n)).$$

Proof. First note that we have chosen the propagator φ so that it is a trivial form, and we have assumed that the morphisms $R \to \Omega^*_{\mathrm{PA}}(M)$ and $R_\partial \to \Omega^*_{\mathrm{PA}}(\partial M)$ factor through the sub-CDGAs of trivial forms. Hence for any graph Γ, $\omega'(\Gamma)$ is a trivial form and can be integrated along the fiber of $p_{U,V}$.

We can now reuse the proof of Proposition 2.3.17. The difference is the description of the decomposition of the fiberwise boundary of $p_{U,V}$ used to show that ω is a chain map through the application of Stokes' formula. This description is very similar to the one implicitly used by [Wil15] (see also [Kon03, Section 5.2.1]) with some variations accounting from the fact that no normalization is done to compactify M (see the discussion before the proof of Lemma 2.3.22). More concretely, the boundary of $\mathsf{SFM}_M(U,V)$ is given by:

$$\partial \mathsf{SFM}_M(U,V) = \bigcup_{T \in \mathcal{BF}'(V)} \mathrm{im}(\circ_T) \cup \bigcup_{(W,T) \in \mathcal{BF}''(U;V)} \mathrm{im}(\circ_{W,T}),$$

where:

$$\begin{aligned} \mathcal{BF}'(V) &:= \{T \subset V \mid \#T \geq 2\}, \\ \mathcal{BF}''(U;V) &:= \{(W,T) \mid W \subset U,\ T \subset V,\ 2 \cdot \#T + \#W \geq 2\}. \end{aligned}$$

Note that in the description of $\partial \mathsf{SFM}_n(U,V)$, there is an additional condition $W \cup T \subsetneq U \cup V$. Indeed in SFM_n the normalization by the affine group prevents the points from becoming infinitesimally close all at once; in SFM_M, no such normalization occurs.

Then the fiberwise boundary of the canonical projection $p_{U,V}$ is given by:

$$\mathsf{SFM}^\partial_M(U,V) = \bigcup_{T \in \mathcal{BF}'(V,J)} \mathrm{im}(\circ_T) \cup \bigcup_{(W,T) \in \mathcal{BF}''(U,I;V,J)} \mathrm{im}(\circ_{W,T}) \subset \mathsf{SFM}_M(U \sqcup I, V \sqcup J),$$

where

$$\begin{aligned} T \in \mathcal{BF}'(V,J) \subset \mathcal{BF}'(V \sqcup J) &\iff \#(T \cap J) \leq 1, \\ (W,T) \in \mathcal{BF}''(U,I;V,J) \subset \mathcal{BF}''(U \sqcup I, V \sqcup J) &\iff V \cap T = \emptyset, \#(U \cap W) \leq 1. \end{aligned}$$

One can then check that the boundary faces of that decomposition correspond to the summands of the differential. □

Definition 3.5.18. Define the **full colored graph complex**

$$\mathrm{fSGC}_R := \mathrm{Tw}\,\mathtt{SGra}_{R,\Omega^*(\partial M)}(\emptyset, \emptyset)[-n].$$

This is the (shifted) CDGA of colored, labeled graphs with only internal vertices. The product is the disjoint union of graphs, thus $\mathrm{fSGC}_R[n]$ is free as an algebra, generated by the graded module $\mathrm{SGC}_R[n]$ of connected graphs. Each $\mathrm{Tw}\,\mathtt{SGra}_R(U, V)$ is a module over the CDGA $\mathrm{fSGC}_R[n]$ by adding connected components.

Definition 3.5.19. The **colored partition function** $Z^S_\varphi : \mathrm{fSGC}_R[n] \to \mathbb{R}$ is the CDGA morphism given by the restriction in empty arity

$$Z^S_\varphi := \omega|_{(\emptyset,\emptyset)} : \mathrm{Tw}\,\mathtt{SGra}_{R,\Omega^*(\partial M)}(\emptyset, \emptyset) \to \Omega^*_{\mathrm{PA}}(\mathtt{SFM}_M(\emptyset, \emptyset)) = \Omega^*_{\mathrm{PA}}(\{*\}) = \mathbb{R}.$$

Let $\mathbb{R}^S_\varphi$ be the one-dimensional $\mathrm{fSGC}_R[n]$-module induced by Z^S_φ.

Remark 3.5.20. Since $\mathrm{fSGC}_R[n]$ is a CDGA, the dual module $\mathrm{SGC}^\vee_R$ is naturally an $\mathtt{hoLie}_n$-algebra. The differential of fSGC_R cannot create more than two connected components, thus $\mathrm{SGC}^\vee_R$ is actually a $\mathtt{Lie}_n$-algebra. The differential blows up vertices (like in $\mathrm{GC}^\vee_n$) and joins pairs of vertices by an edge, while the Lie bracket joins two graphs by an edge. The algebra morphism Z^S_φ is uniquely determined by its restriction z^S_φ to SGC_R, which can be seen as a Maurer–Cartan element in the $\mathtt{Lie}_n$-algebra $\mathrm{SGC}^\vee_R$.

Definition 3.5.21. The **reduced colored labeled graph comodule** $\mathtt{SGraphs}^{c_\varphi, z^S_\varphi}_{R,\Omega^*(\partial M)}$ is the bisymmetric collection given in each arity by:

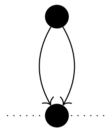
$$\mathtt{SGraphs}^{c_\varphi, z^S_\varphi}_{R,\Omega^*(\partial M)}(U, V) := \mathbb{R}^S_\varphi \otimes_{\mathrm{fSGC}_R[n]} \mathrm{Tw}\,\mathtt{SGra}_{R,\Omega^*(\partial M)}(U, V).$$

Proposition 3.5.22. *The bisymmetric collection* $\mathtt{SGraphs}^{c_\varphi, z^S_\varphi}_{R,\Omega^*(\partial M)}$ *forms a Hopf right comodule over* $\mathtt{SGraphs}_n$*, and the map* $\omega : \mathrm{Tw}\,\mathtt{SGra}_{R,\Omega^*(\partial M)} \to \Omega^*_{\mathrm{PA}}(\mathtt{SFM})$ *factors through a Hopf right comodule morphism:*

$$\omega : (\mathtt{SGraphs}^{c_\varphi, z^S_\varphi}_{R,\Omega^*(\partial M)}, \mathtt{SGraphs}_n) \to (\Omega^*_{\mathrm{PA}}(\mathtt{SFM}_M), \Omega^*_{\mathrm{PA}}(\mathtt{SFM}_n)).$$

Proof. Identical to the proof of Proposition 2.3.31. □

We now show that our comodule is quasi-isomorphic to a simpler one, in the spirit of Corollary 3.4.7. Recall the circular graph from Figure 3.4.1. We will need an explicit name for the circular graph of length 1 in GC_{n-1}, say:

$\ell_1 :=$  (3.5.23)

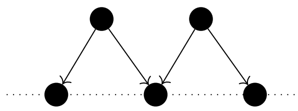
Lemma 3.5.24. *The Maurer–Cartan element $c_\varphi \in \mathrm{SGC}_n^\vee \otimes \Omega^*_{\mathrm{triv}}(\partial M)$ is gauge equivalent to*

$$c_M := c_0 \otimes 1 + \ell_1^\vee \otimes e_{\partial M},$$

where:

- *$c_0 \in \mathrm{SGC}_n^\vee$ is the linear part of c (Definition 3.4.2);*
- *$\ell_1 \in \mathrm{SGC}_n$ is the circular graph of length 1 in GC_{n-1} (Equation (3.5.23));*
- *$e_{\partial M} := \chi(\partial M) \cdot \mathrm{vol}_{\partial M}$ is the Euler class of ∂M (and $\mathrm{vol}_{\partial M} = g_\partial(\mathrm{vol}_{R_\partial})$ where $\mathrm{vol}_{R_\partial}$ is a fixed cocycle satisfying $f_\partial(\mathrm{vol}_{R_\partial}) = \mathrm{vol}_{B_\partial}$).*

Proof. Let us first show that c_φ agrees with c_M on circular graphs, i.e. that c_φ vanishes on circular graphs except for ℓ_1, on which it is equal to e_M. The aerial circular graphs which produce nontrivial classes have odd length,[6] hence they contain a vertex with in and out valences of 1. One can construct the propagator ψ on $D = M \cup_{\partial M} \bar{M}$ (see Section 3.5.1) so that φ exhibits symmetry properties that make c_φ vanish on these graphs, by an argument similar to Kontsevich's trick [Kon94, Lemma 2.1]. Moreover, using a similar argument, we see that c_φ vanishes on the circular graphs from GC_{n-1}, because apart from the first one, they all contain a copy of the following graph (which represent a "bivalent" terrestrial vertex, the "outside" vertices may be the same one):



Thus when computing the integral defining c, one can consider the automorphism of $\mathsf{SFM}_n(\{u_1, u_2, v\}, \emptyset)$ which maps v to its symmetric with respect to the barycenter of u_1 and u_2 inside $\mathbb{R}^{n-1} \times \{0\}$, which changes the sign of the integral. The only exception is ℓ_1, for which we can explicitly compute $c_M(\ell_1) = e_M$ using the fourth property of φ in Proposition 3.5.8 and the standard fact that the Euler class $e_{\partial M}$ is the image of the diagonal class of ∂M under the multiplication map.

[6] Depending on the parity of n, they are congruent to either 1 or 3 modulo 4.

It then follows that $c_\varphi - c_M$ is a Maurer–Cartan element in the twisted Lie algebra $\mathrm{SGC}_n^{\vee,c_0} \otimes \Omega^*(\partial M)$. According to Proposition 3.4.4, if we forget about circular graphs the cohomology of $\mathrm{SGC}_n^{\vee,c_0}$ is concentrated below degree $-(n-1)$, and the cohomology of ∂M below degree $n-1$. It follows that $c_\varphi - c_M$ lives in a Lie subalgebra of $\mathrm{SGC}_n^{\vee,c_0} \otimes \Omega^*(\partial M)$ whose cohomology vanishes in positive degrees. Thus, by obstruction theory, $c_\varphi - c_M$ is gauge trivial and we have proved the proposition. $\square$

Recall the cooperad $\mathsf{SGraphs}_n^{c_0}$ from Definition 3.4.6. We can define a Hopf right comodule $\mathsf{SGraphs}_{R,\Omega^*(\partial M)}^{c_M,z_\varphi^S}$ over this cooperad, by twisting $\mathsf{SGra}_{R,\Omega^*(\partial M)}$ with respect to c_M instead of c_φ.

Corollary 3.5.25. *There is zigzag of quasi-isomorphisms of Hopf right comodules:*

$$(\mathsf{SGraphs}_{R,\Omega^*(\partial M)}^{c_\varphi,z_\varphi^S}, \mathsf{SGraphs}_n) \xleftarrow{\sim} \cdot \xrightarrow{\sim} (\mathsf{SGraphs}_{R,\Omega^*(\partial M)}^{c_M,z_\varphi^S}, \mathsf{SGraphs}_n^{c_0}). \quad \square$$

Moreover, we can now get rid of the $\Omega^*(\partial M)$ and label the terrestrial vertices by elements of R_∂ instead. This was not possible before vecause c_φ could produce forms that were not in the image of $g_\partial : R_\partial \xrightarrow{\sim} \Omega^*(\partial M)$, but this is not the case for $c_M \otimes 1$. We thus define a Hopf right comodule $(\mathsf{SGraphs}_{R,R_\partial}^{c_M,z_\varphi^S}, \mathsf{SGraphs}_n^{c_0})$, and we obtain:

Lemma 3.5.26. *There is a quasi-isomorphism of Hopf right comodules induced by $g_\partial : R_\partial \xrightarrow{\sim} \Omega^*(\partial M)$:*

$$(\mathsf{SGraphs}_{R,R_\partial}^{c_M,z_\varphi^S}, \mathsf{SGraphs}_n^{c_0}) \xrightarrow{\sim} (\mathsf{SGraphs}_{R,\Omega^*(\partial M)}^{c_0,z_\varphi^S}, \mathsf{SGraphs}_n^{c_0}). \quad \square$$

3.5.3 Labeled graphs and proof of Theorem D

We now introduce a variant $\mathsf{Graphs}_R^{z_\varphi}$ of the graph comodule from Chapter 2. Recall that $j^*\varphi = \varphi|_{\mathsf{SFM}_M(\{2\},\{1\})}$ is equal to $(g_\partial \otimes g)(\sigma_R)$, where $\sigma_R = \sum \sigma' \otimes \sigma'' \in R_\partial \otimes R$ was defined in Proposition 3.5.8.

Definition 3.5.27. The CDGA of reduced R-labeled graphs $\mathsf{Graphs}_R^{z_\varphi}(V)$ on the finite set V is spanned by directed graphs with external (corresponding to V) vertices and internal vertices, with all the vertices labeled by R. The algebra structure glues graphs along external vertices, and the comodule structure collapses subgraphs. A connected component with only internal vertices is identified with the real number given by the partition function Z_φ^S, by viewing the component as having no terrestrial vertices. The differential has three parts:

- Splitting any edge and multiplying the labels of the endpoints by Δ_R;
- Contracting an edge between an internal vertex and another vertex, multiplying the labels;
- Removing one internal vertex u. Let $x \in R$ be the label of u, and let $v_1, \dots, v_k$ be the vertices connected to u, with respective labels $y_1, \dots, y_k \in R$. Then in the differential, the vertex u and its incident edges are removed, the label of v_i becomes $y_i \sigma_i''$, and the new graph is multiplied by $\varepsilon_\partial(\rho(x)\sigma_1' \dots \sigma_k') \in \mathbb{R}$. See Figure 3.5.4 for an example.

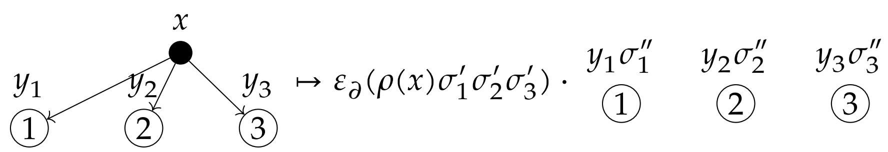


Figure 3.5.4: The third part of the differential in $\mathtt{Graphs}_R^\varphi$

Roughly speaking, in this third part of the differential, the removed vertex becomes infinitesimally close to the boundary of M. Each edge thus becomes $j^*\varphi = \varphi|_{\mathsf{SFM}_M(\{1\},\{2\})} = (g_\partial \otimes g)(\sigma_R)$, and we multiply the labels accordingly. The vertex u is then alone in its connected component and is thus identified with the integral of its label.

For each finite set of external vertices V, there is a map $\nu_R : \mathtt{SGraphs}_{R,R_\partial}^{c_M, z_\varphi^S}(\emptyset, V) \to \mathtt{Graphs}_R^{z_\varphi}(V)$ defined as follows. Given some graph $\Gamma \in \mathtt{SGraphs}_{R,R_\partial}^{c_M, z_\varphi^S}(\emptyset, V)$, we remove all the edges from an aerial vertex to a terrestrial vertex and we multiply their endpoints by $\sigma_R \in R_\partial \otimes R$. We then obtain a graph where all the terrestrial vertices are zero-valent, and we identify those with the real number given by the integral over ∂M of their labels. In this way, we obtain an element of $\mathtt{Graphs}_R^{z_\varphi}(V)$. This is a chain map, the part of the differential described in Figure 3.5.4 being induced by Figure 3.5.3.

Proposition 3.5.28. *The symmetric collection* $\mathtt{Graphs}_R^\varphi$ *is a Hopf right comodule over* $\mathtt{Graphs}_n$. *The maps*

$$\nu_R : \mathtt{SGraphs}_R^{c_M, z_\varphi^S}(\emptyset, -) \xrightarrow{\sim} \mathtt{Graphs}_R^{z_\varphi}$$

define a quasi-isomorphism of Hopf right $\mathtt{Graphs}_n$*-comodules.*

Proof. We use an argument similar to the one of Section 2.4.2. Filter both complexes by the number of edges minus the number of vertices. The only part of the differential which remains on the $\mathtt{E}^0$ page is the contraction of aerial edges and the contraction of a subgraph into a terrestrial vertex induced by c_M, which is nonzero on trees only (see Definition 3.4.2 and Lemma 3.5.24, the summand induced by $\ell_1 \otimes e_{\partial M}$ vanishes because it strictly decreases the filtration).

It follows that both complexes split as a direct sum in terms of connected components, just like in Lemma 2.4.21. We can now reuse the same "trick" we used for the proof of Lemma 2.4.27, to show that in cohomology, there is only one label for each connected component. We thus reduce to the morphism $\nu : \mathtt{SGraphs}_n^{c_0}(\emptyset, V) \to \mathtt{Graphs}_n(V)$ from Proposition 3.4.10 (tensored with the identity of R for each connected component). Since that morphism is a quasi-isomorphism, we obtain that the induced morphism on the $\mathtt{E}^1$ page is an isomorphism. It follows by standard spectral sequence arguments (the filtration is bounded below for a fixed V) that the morphism of the proposition is a quasi-isomorphism. □

We would now like to connect $\mathtt{Graphs}_R^{z_\varphi}(V)$ with $\tilde{\mathtt{G}}_A(V)$.

Definition 3.5.29. The **full labeled graph complex** fGC_R is the CDGA of aerial graphs with only internal vertices, the differential being defined in a manner similar to Definition 3.5.27. The **partition function** $\mathrm{Z}_\varphi : \mathrm{fGC}_R[n] \to \mathbb{R}$ is induced by Z_φ^S. Let $\mathbb{R}_\varphi$ be the one-dimensional $\mathrm{fGC}_R[n]$-module induced by Z_φ.

The (shifted) commutative algebra fGC_R is free, generated by the submodule GC_R of connected graphs. The differential of fGC_R turns $\mathrm{GC}_R^\vee$ into a $\mathtt{Lie}_n$-algebra up to homotopy (see Remark 3.5.20, and note that the part of the differential that removes one vertex may produce more than two connected components, so this is not a $\mathtt{Lie}_n$-algebra), and Z_φ defines a Maurer–Cartan element $\mathrm{z}_\varphi \in \mathrm{GC}_R^\vee$.

We can reuse the counting argument from Proposition 2.3.35 (and the last property of the propagator in Proposition 3.5.8) to show that the partition function vanishes on any graph with no bivalent vertex labeled by 1_R. We can then define $\mathrm{Z}_\varepsilon : \mathrm{fGC}_R \to \mathbb{R}$ to be the algebra morphism which vanishes on all connected graphs γ except those with a single vertex. If that vertex is labeled by $x \in R$ we set $\mathrm{Z}_\varepsilon(\gamma) = \int_M g(x)$.

Unfortunately, we still cannot prove directly that $\mathrm{Z}_\varphi = \mathrm{Z}_\varepsilon$; however, we can follow the discussion at the beginning of Section 2.4. Let $\Omega^*(\Delta^1) = S(t, dt)$ be the algebra of polynomials forms on Δ^1, a path object in the category of CDGAs. We then obtain:

Proposition 3.5.30. *There exists a homotopy $h : \mathrm{fGC}_R[n] \to \Omega^*(\Delta^1)$ such that the following diagram commutes:*

$$\begin{tikzcd} & \mathrm{fGC}_R[n] \arrow[dl, "Z_\varphi"'] \arrow[d, "h"] \arrow[dr, "Z_\varepsilon"] & \\ \mathbb{R} & \Omega^*(\Delta^1) \arrow[l, "d_1", "\sim"'] \arrow[r, "d_0"', "\sim"] & \mathbb{R} \end{tikzcd}$$
□

The reduced labeled graph comodule can be viewed as a tensor product $\mathbb{R}_\varphi \otimes_{\mathrm{fGC}_R[n]} \mathrm{Tw}\,\mathsf{Gra}_R(V)$. We can then define $\mathsf{Graphs}'_R := \Omega^*(\Delta^1)_h \otimes_{\mathrm{fGC}_R[n]} \mathrm{Tw}\,\mathsf{Gra}_R$ and $\mathsf{Graphs}_R^{Z_\varepsilon} := \mathbb{R}_\varepsilon \otimes_{\mathrm{fGC}_R[n]} \mathrm{Tw}\,\mathsf{Gra}_R$. Both symmetric collections define Hopf right Graphs_n-comodules.

Proposition 3.5.31. *We have a zigzag of quasi-isomorphisms of comodules:*

$$\mathsf{Graphs}_R^{Z_\varepsilon} \xleftarrow{\sim} \mathsf{Graphs}'_R \xrightarrow{\sim} \mathsf{Graphs}_R^{Z_\varphi}.$$

Proof. The $\mathrm{fGC}_R[n]$-module $\mathrm{Tw}\,\mathsf{Gra}_R(V)$ is quasi-free and is equipped with a good filtration, hence $\mathrm{Tw}\,\mathsf{Gra}_R(V) \otimes_{\mathrm{fGC}_R[n]} (-)$ preserves quasi-isomorphisms. The claim now follows from the previous proposition. □

We can now do the same construction as $\mathsf{Graphs}_R^{Z_\varepsilon}$ but with B replacing R, Δ_B (from Definition 3.5.5) replacing Δ_R, σ_B replacing σ_R, and taking as "partition function" the map which sends a graph with a single vertex labeled by b to $\varepsilon_B(b)$ and all other connected graphs to zero. We obtain a Hopf right Graphs_n-comodule Graphs_B, where graphs with internal components containing at least two vertices vanish, and an isolated internal vertex labeled by $b \in B$ is identified with $\varepsilon_B(b)$.

Finally, we can consider a quotient Graphs_A of Graphs_B, where we apply the projection $\pi : B \to A$ to all the remaining labels.

It is not necessarily the case that the two morphisms $\varepsilon, \varepsilon' : \mathrm{cone}(\rho) \to \mathbb{R}[-n+1]$, defined respectively by the composites $(\int_M g(-), \int_{\partial M} g_\partial(-))$ and $(\varepsilon_B f, \varepsilon_{B_\partial} f_\partial)$, are equal. Nevertheless, up to rescaling ε (which induces an automorphism of $\mathsf{Graphs}_R^{Z_\varepsilon}$), they induce the same map up to quasi-isomorphism, $H^{n-1}(\mathrm{cone}(\rho))$ being one-dimensional. Since all cochain complexes are fibrant and cofibrant, we obtain a diagram:

$$\begin{tikzcd} & \mathrm{cone}(\rho) \arrow[dl, "\varepsilon"'] \arrow[d, "\varepsilon_h"] \arrow[dr, "\varepsilon'"] & \\ \mathbb{R}[-n+1] & P[-n+1] \arrow[l, "d_1", "\sim"'] \arrow[r, "d_0"', "\sim"] & \mathbb{R}[-n+1] \end{tikzcd} \qquad (3.5.32)$$

where P is the standard path object for the base field in the category of cochain complexes. The chain map ε_h is uniquely determined by a graded map $h : \mathrm{cone}(\rho) \to \mathbb{R}[-n+1]$ satisfying $\varepsilon(x) - \varepsilon'(x) = h(dx)$.

Since the product of fGC_R is merely given by the disjoint union of graphs, this yield a homotopy between the two morphisms $\mathrm{fGC}_R[n] \to \mathbb{R}$ induced. Similarly to Proposition 3.5.31, we then obtain a zigzag of quasi-isomorphisms:

$$\mathtt{Graphs}_R^{z_\varepsilon} \xleftarrow{\sim} \mathtt{Graphs}_R'' \xrightarrow{\sim} \mathtt{Graphs}_R^{z'_\varepsilon}. \tag{3.5.33}$$

Now ε' is compatible with $(\varepsilon_B, \varepsilon_{B_\partial})$, we obtain the following proposition:

Proposition 3.5.34. *We have quasi-isomorphisms of Hopf right* $\mathtt{Graphs}_n$*-comodules*

$$\mathtt{Graphs}_R^{z'_\varepsilon} \xrightarrow{f_*} \mathtt{Graphs}_B \xrightarrow{\pi_*} \mathtt{Graphs}_A$$

which apply $f : R \to B$ (resp. $\pi : B \to A$) to all the labels.

Proof. Filter the graph comodules by the number of edges. On each E^0 page, only the differentials from R (resp. B, A) remain. Since f and π are a quasi-isomorphism, it follows that the induced morphisms on E^1 pages is an isomorphism. Standard spectral sequence arguments then imply that f_* and π_* are quasi-isomorphisms. □

We can now define a morphism $\mathtt{Graphs}_A \to \tilde{\mathtt{G}}_A$ which maps all graphs with internal vertices to zero, and which sends an edge $e_{vv'}$ between external vertices to $\tilde{\omega}_{vv'}$. Then we can mimic the proof of Proposition 2.4.17 in a straightforward way (note that as soon as we filter by #edges − #vertices the perturbed relations of $\tilde{\mathtt{G}}_A$ become the usual relations of $\mathtt{G}_A$):

Proposition 3.5.35. *This defines a quasi-isomorphism of right Hopf comodules:*

$$(\mathtt{Graphs}_A, \mathtt{Graphs}_n) \to (\tilde{\mathtt{G}}_A, \mathtt{e}_n^\vee).$$

We can also copy the proof of Proposition 2.4.31 and use Theorem 3.3.16 to obtain:

Proposition 3.5.36. *The morphism* $\omega : \mathtt{SGraphs}_R^{c_\varphi, z_\varphi^S}(\emptyset, -) \to \Omega^*_{\mathrm{PA}}(\mathtt{SFM}_M(\emptyset, -))$ *is a quasi-isomorphism.*

Finally we can summarize this section as:

Theorem 3.5.37 (Precise version of Theorem D). *Let M be a simply connected, smooth, compact manifold with boundary of dimension at least 5, and assume either that it admits a surjective pretty model or that $\dim M \geq 7$ so that it admits a PLD. Let A be the CDGA model built from the resulting PLD model.*

Then the symmetric collection of CDGAs $\tilde{\mathsf{G}}_A$ is quasi-isomorphic to $\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M(\emptyset, -))$. If moreover M is framed, then the Hopf right comodule $(\tilde{\mathsf{G}}_A, \mathsf{e}_n^\vee)$ is quasi-isomorphic to $(\Omega^*_{\mathrm{PA}}(\mathsf{SFM}_M(\emptyset, -)), \Omega^*_{\mathrm{PA}}(\mathsf{FM}_n))$. ☐

Corollary 3.5.38 (Corollary E). *The real homotopy type of configuration spaces on a simply connected, smooth manifold M with simply connected boundary only depends on the real homotopy type of the manifold as soon as:*

- *either* $\dim M \geq 5$ *and M admits a surjective pretty model;*
- *or* $\dim M \geq 7$.

Example 3.5.39. We can apply this to $M = D^n$, using the surjective pretty model from Example 3.1.16. Recall that in this case, $A = \mathbb{R}$, $\Delta_A = 0$, $\sigma_A = 0$, and $\tilde{\mathsf{G}}_A$ is isomorphic to $\mathsf{e}_n^\vee$ as a Hopf right comodule over itself (see Example 3.3.26). We then "recover" the already known fact that $\mathsf{SFM}_{D^n}(\emptyset, -)$ is (Hopf) formal as a right FM_n-module (though that "proof" is of course circular).

The Hopf right Graphs_n-comodule Graphs_A is isomorphic to Graphs_n seen as a comodule over itself. The augmentation $\varepsilon_B : B \to \mathbb{R}[-n+1]$ yields a Maurer–Cartan element z_ε in the abelian $\mathtt{hoLie}_n$-algebra $\mathrm{GC}_A^\vee$ (see Remark 3.5.20), given in the dual basis by the graph with a single vertex labeled by vol_n. The twisted $\mathtt{hoLie}_n$-algebra $\mathrm{GC}_A^{\vee, \mathsf{z}_\varepsilon}$ is then isomorphic to $\mathrm{GC}_n^\vee$ from Section 3.1.5.[7]

3.5.4 Proof of Theorem F

Proposition 3.5.40. *Assume that $n = \dim M \geq 5$. Then for all finite sets U and V, $H^*(\mathsf{SGraphs}_R^{c_\varphi, z_\varphi^S}(U, V))$ has the same dimension in each degree as*

$$H^*(\mathsf{SFM}_M(U, V)) \cong H^*(\mathrm{Conf}_U(\partial M)) \otimes H^*(\mathrm{Conf}_V(M)).$$

Proof. We use a spectral sequence argument similar to the proof in [Wil15, Section 5]. The assumption about the dimension of M allows us to apply Theorem 3.5.37 to M and Theorem C to ∂M.

First filter $\mathsf{SGraphs}_R^{c_\varphi, z_\varphi^S}(U, V)$ by the number of edges minus the number of vertices as in the proof of the previous proposition. We get as an E^0 page the graded module $\mathsf{SGraphs}_R(U, V)$ together with the differential which contracts

[7] The twisting ensures that dead ends are not contractible, see the proof of Proposition 2.4.13.

edges connected to an aerial internal vertex. Filtering this new complex by the number of vertices that are not of the type "bivalent, internal, aerial, and labeled by 1_R" as in the proof of [Wil15, Section 5], and using the previous proposition, we obtain that E^0 is quasi-isomorphic to its quotient $\mathcal{Q}(U, V)$ in which:

- graphs containing bivalent internal aerial vertices labeled by 1_R other than those appearing in a zigzag of the type shown in Figure 3.5.5 are set to zero;
- graphs with edges between aerial and terrestrial vertices other than those appearing in a zigzag of the type shown in Figure 3.5.5 are set to zero;
- an edge $e_{vv'}$ between aerial vertices is identified with $(-1)^n e_{v'v}$.

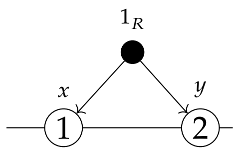


Figure 3.5.5: A zigzag which represents an edge in $\mathsf{Graphs}_{R_\partial}(U)$.

We may identify the underlying graded module of $\mathcal{Q}(U, V)$ with the tensor product $\mathsf{Graphs}_R(V) \otimes \mathsf{Graphs}_{R_\partial}(U)$. The induced differential d^0 contracts edges in $\mathsf{Graphs}_R(V)$ (but not $\mathsf{Graphs}_{R_\partial}(V)$). We obtain that the underlying graded module of the E^1 page is isomorphic to

$$H^*(\mathsf{Graphs}_R(V), d^0) \otimes \mathsf{Graphs}_{R_\partial}(V).$$

The differential d^1 is induced by the part of the differential which decreases the filtration by exactly 1. By comparing with the proof of Theorem 3.5.37 (and by extension with the proof of Theorem C), we obtain that the complex

$$(H^*(\mathsf{Graphs}_R(V), d^0), d^1)$$

is isomorphic to $\mathsf{G}_A(V)$, which has the same Betti numbers as $\mathrm{Conf}_V(M)$ by Theorem 3.3.16. Moreover, by inspection, d^1 is given on $\mathsf{Graphs}_{R_\partial}(U)$ by the contraction of edges (and the multiplication of labels). Applying Theorem C to ∂M, we thus obtain that the E^2 page of the spectral sequence is isomorphic to:

$$H^*(\mathsf{G}_A(V)) \otimes \mathsf{G}_{B_\partial}(U) \cong H^*(\mathrm{Conf}_V(M)) \otimes \mathsf{G}_{B_\partial}(U).$$

Since any remaining cocycle is represented by a cocycle in $\mathsf{SGraphs}_R^{c_\varphi, z_\varphi^S}(U, V)$, we obtain that the spectral sequence abuts at this stage. □

Proposition 3.5.41. *Assume that $n = \dim M \geq 5$. Then the morphism*

$$\omega : \mathsf{SGraphs}_{R,\Omega^*(\partial M)}^{c_\varphi, z_\varphi^S}(U, V) \to \Omega_{\mathrm{PA}}^*(\mathsf{SFM}_M(U, V))$$

is a quasi-isomorphism of CDGAs for all U, V.

Proof. We know that $H^*(\mathsf{SFM}_M(U, V))$ is isomorphic to the tensor product

$$H^*(\mathrm{Conf}_V(M)) \otimes H^*(\mathrm{Conf}_U(\partial M)).$$

We can then use the previous proposition and an inductive argument similar to the proof of Proposition 2.4.31 to obtain the result. Indeed we can represent any cocycle in $H^*(\mathrm{Conf}_V(M))$ by a cocycle in $\mathsf{Graphs}_R^\varphi(V)$, and each cocycle in $H^*(\mathrm{Conf}_U(\partial M))$ by a cocycle in $\mathsf{Graphs}_{R_\partial}^{\varphi_\partial}(U)$, and then take the disjoint union of the resulting graphs. □

We can now bundle everything together into a theorem, with a summary of our hypotheses.

Theorem 3.5.42 (Precise version of Theorem F)**.** *Let M be a smooth, simply connected manifold with a simply connected boundary, of dimension $n \geq 5$. Suppose also that M admits a surjective pretty model, or that $n \geq 7$ so that it admits a PLD model as in Equation (3.2.17).*

Then for each U, V, then the CDGA $\mathsf{SGraphs}_{R,R_\partial}^{c_M, z_\varphi^S}(U, V)$ is quasi-isomorphic to $\Omega_{\mathrm{PA}}^*(\mathsf{SFM}_M(U, V))$, and this is compatible with the symmetric group actions.

If M is framed, then together with Willwacher's morphism [Wil15], this defines a quasi-isomorphism of right Hopf comodules

$$(\mathsf{SGraphs}_{R,R_\partial}^{c_M, z_\varphi^S}, \mathsf{SGraphs}_n^{c_0}) \simeq (\Omega_{\mathrm{PA}}^*(\mathsf{SFM}_M), \Omega_{\mathrm{PA}}^*(\mathsf{SFM}_n)).$$

It would be an interesting question to know how much z_φ^S can be simplified.

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Index