Homotopy Prefactorization Algebras

Najib Idrissi[*] Eugene Rabinovich[†]

April 2023

We apply the theory of operadic Koszul duality to provide a cofibrant resolution of the colored operad whose algebras are prefactorization algebras on a fixed space M. This allows us to describe a notion of prefactorization algebra up to homotopy as well as morphisms up to homotopy between such objects. We make explicit these notions for several special M, such as certain finite topological spaces, or the real line.

Contents

[*]Université Paris Cité and Sorbonne Université, CNRS, IMJ-PRG, F-75013 Paris, France.
[†]University of Notre Dame, Deparment of Mathematics

1 Introduction

Prefactorization algebras and factorization algebras—their cousins satisfying descent—are objects meant to model the structure present in the observables of quantum field theory (see [CG17; CG21]). A prefactorization algebra on a topological space M is a precosheaf $\mathcal{F}$ on M equipped with extra structure maps that can "glue" elements of $\mathcal{F}(U_1), \ldots, \mathcal{F}(U_n)$ (for pairwise disjoint open subsets $U_i \subset V$) into a single element of $\mathcal{F}(V)$ (see Definition 2.2). (Pre-)factorization algebras with additional properties give rise to familiar algebraic objects, such as vertex algebras or E_n-algebras. Some tools to study the homotopy theory of prefactorization algebras have been given by Carmona–Flores–Muro [CFM21], including a model structure on the category of prefactorization algebras on M such that the bifibrant objects in this category are (a subclass of the) locally constant factorization algebras on M.

In the current paper, we provide a complementary perspective, in which we focus on computational tools for the study of the ∞-category of prefactorization algebras. Namely, we describe a notion of "prefactorization algebra up to homotopy", a notion of "∞-morphism of homotopy prefactorization algebras", and a homotopy transfer theorem for such algebras. Our main technique is the Koszul theory for operads, of which we provide an overview in 2.1 for those readers unfamiliar with this toolkit.

We can state the main results of this paper as follows:

Theorem A (See Definition 2.1, Theorem 3.4)**.** Given a manifold M, there is a *Koszul* quadratic-linear colored operad Disj_M whose category of algebras is the category of prefactorization algebras on M.

Theorem A implies that there is a relatively simple cofibrant resolution hoDisj_M of Disj_M, and we make this cofibrant resolution explicit. Furthermore, we make explicit the category of algebras over this operad.

Theorem B (See Proposition 4.1)**.** Given a manifold M, algebras over the Koszul resolution $\mathrm{hoDisj}_M = \Omega(\mathrm{Disj}_M^{\text{i}})$ are called homotopy prefactorization algebras on M. Such an algebra is given by a collection $\mathcal{A} = \{\mathcal{A}(U)\}_{U \subseteq M}$ indexed by open subsets of M, equipped with maps:

$$\mu_{\mathcal{U}} : \mathcal{A}(U_{11}) \otimes \cdots \otimes \mathcal{A}(U_{k1}) \to \mathcal{A}(U_{1s_1} \sqcup \cdots \sqcup U_{ks_k})$$

for every collection $\mathcal{U} = (U_{11} \subset \cdots \subset U_{1s_1}, \ldots, U_{k1} \subset \cdots \subset U_{ks_k})$ (with pairwise disjoint U_{js_j}), satisfying several compatibility conditions (most notably Equation (4.1)).

The exact relations satisfied by the $\mu_{\mathcal{U}}$ are not tremendously important; the important thing is that Proposition 4.1 makes these relations completely explicit. Finally, we discuss the homotopy transfer theorem for hoDisj_M algebras. Readers may be familiar with the idea that, given a dg associative algebra A, its cohomology $H^\bullet(A)$ is also a dg associative algebra; however, there is not necessarily a map of dg associative algebras $H^\bullet(A) \to A$. Instead, one must consider the data $H^\bullet(A)$ together with its *Massey products*, which are at-least-trilinear operations on $H^\bullet(A)$. These Massey products endow

$H^\bullet(A)$ with the structure of an A_∞-algebra, and this A_∞-algebra is equivalent to A in a suitable sense. An analogous story is true for algebras over any Koszul operad, and in fact, as a consequence of Theorem B, we find:

Theorem C (See Proposition 4.4)**.** Given a prefactorization algebra $\mathcal{A}$ on M and deformation retractions

$$\mathcal{B}(U) \underset{p_U}{\overset{i_U}{\rightleftarrows}} \mathcal{A}(U) \circlearrowleft h_U$$

for all open subsets $U \subset M$, then the collection $\{\mathcal{B}(U)\}_{U \in \mathrm{Opens}(M)}$ has a hoDisj-algebra structure and the maps i_U can be extended into an ∞-morphism of hoDisj-algebras.

1.1 Outline

The paper is organized as follows. In Section 2.2, we present the operad Disj_M, and we give a simple generators-and-relations presentation thereof. Because we imagine a mixed audience of operadic and factorization-algebraic readers, we start the section with gentle introductions for each type of reader to the other field (cf. Sections **??** and 2.1). Next, in Section 3, we prove Theorem C. The reader interested in "the punchline" may skip straight to Section 4, in which we make explicit the definition of a hoDisj_M-algebra and ∞-morphism thereof. We also state Theorem C in Section 4. Finally, in Section 5, we describe hoDisj-algebras on some finite spaces, including the empty set, the one-point space, and the Sierpiński space. We also extend a theorem of Costello–Gwilliam [CG17] concerning the factorization enveloping algebra $\tilde{\mathcal{F}}_{\mathfrak{g}}$, which is a prefactorization algebra on $\mathbb{R}$.

1.2 Future Directions

The Koszul property (and its proof) of the operad Disj_M opens up several avenues for future research.

The first is the study of the deformation complex of prefactorization algebras, a kind of "homology theory" naturally defined for algebras over (Koszul) operads. Prefactorization algebras have properties reminiscent of commutative algebras (see Section 5). Locally constant unital prefactorization algebras on $M = \mathbb{R}^n$ are equivalent to E_n-algebras, i.e., homotopy associative and commutative algebras. The deformation complex of an E_n-algebra satisfies the Hochschild–Kostant–Rosenberg (HKR) theorem: it splits as the (symmetric) algebras of (shifted) polyvector fields on $\mathbb{R}^n$. For $n = 1$, this is the classical HKR theorem [HKR62]; for higher n, this is due to [CW15]. It is natural to wonder whether the deformation complex of a prefactorization algebra on M splits as the algebras of polyvector fields on M. Theorem C provides a tool to study this question, as it can be used to give an explicit description of the deformation complex of a prefactorization algebra on M.

The second avenue is the relationship with known categories of prefactorization algebras. As mentioned above, (locally constant) prefactorization algebras on the real line $M = \mathbb{R}$ are equivalent to homotopy associative algebras (i.e., E_1-algebras or A_∞-algebras). Given a strictly associative algebra, it is easy to produce a prefactorization algebra on $\mathbb{R}$, i.e., a $\mathrm{Disj}_\mathbb{R}$-algebra. However, given a specific A_∞-algebra, it is not clear how to construct a prefactorization algebra on $\mathbb{R}$ explicitly. Using our methods, we hope to be able to describe an explicit equivalence that produces a $\mathrm{hoDisj}_\mathbb{R}$-algebra from an A_∞-algebra in a way that covers the functors which turns an associative algebra into a $\mathrm{Disj}_\mathbb{R}$-algebra.

Finally, one is generally interested in prefactorization algebras that are equipped with more structure. For example, in the deformation quantization approach to quantum field theory, $\mathbb{P}_0$-prefactorization algebras—whose values are shifted Poisson algebras and whose structure maps are morphisms of Poisson algebras—are of particular interest (see the introduction of [CG17]). However, the natural extension of the presentation of Disj_M to $\mathbb{P}_0$-prefactorization algebras is not quadratic-linear. Indeed, the property of being a morphism of algebras is cubic. This failure can be fixed e.g., by introducing new generators to the presentation, but the computational difficulty quickly becomes prohibitive. Nevertheless, a resolution of $\mathbb{P}_0$-prefactorization algebras up to homotopy would be a valuable tool.

1.3 Conventions

- We fix a manifold M throughout, except possibly in Section 5. We may or may not make M explicit in the notation; for example, Opens_M and Opens will both refer to the poset (or category, or colored operad) of open subsets of M.
- Throughout, we fix the ground field to be $\mathbb{R}$, although the results presented here apply equally well for any field of characteristic zero.
- Fixing a finite set X whose elements are $x_1, \dots, x_N$, we let $\mathbb{R}\{x_1, \dots, x_N\}$ denote the free vector space on X.
- The symbol $\mathbb{S}_k$ denote the symmetric group on k letters.
- A rooted tree T is a set V of vertices, a pointed set (H, r) of half-edges, a map $\mathrm{inc} : H \to V$ which takes each half-edge to the vertex on which it is incident, and an involution $\sigma : H \to H$ such that $\sigma(r) = r$. The orbits of σ are called "edges"; the orbits of cardinality two are called "internal edges", r is called "the root", and all other fixed points of σ are called the "leaves" of T. When we draw a tree, we draw it with the leaves at the top and the root at the bottom. In the operadic analogy, the trees represent compositions from top to bottom.
- The symbol $\mathbb{S}_k$ denotes the symmetric group on k letters.
- By a(n Opens-colored) $\mathbb{S}$-module, we mean a collection $\{E(k)\}_{k\geq 0}$ of vector spaces such that each $E(k)$ is a right $\mathbb{S}_k$-module. Furthermore, each $E(k)$ carries a decomposition
$$E(k) = \bigoplus_{U_1,\dots,U_k,V\in\mathrm{Opens}} E\begin{pmatrix} V \\ U_1, \dots, U_k \end{pmatrix}$$

such that the action of $\sigma \in \mathbb{S}_k$ on $E(k)$ is determined by maps

$$E\begin{pmatrix} V \\ U_1, \dots, U_k \end{pmatrix} \to E\begin{pmatrix} V \\ U_{\sigma^{-1}(1)}, \dots, U_{\sigma^{-1}(k)} \end{pmatrix}.$$

- If E is a graded Opens-colored $\mathbb{S}$-module, $E[1]$ is the graded Opens-colored $\mathbb{S}$-module whose degree p components are the degree $p+1$ components of E. $E\{1\}$ is the following $\mathbb{S}$-module:

$$E\{1\}(k) = E(k) \otimes \mathrm{sgn}_k[k-1].$$

  If E is an operad or cooperad, $E[1]$ is not necessarily also one, but $E\{1\}$ is.
- Most of our notation matches that of we use cohomological grading instead of homological grading and 2) we use the symbol $E\{1\}$ to denote what in the cited reference is referred to as $\mathscr{S}^{-1}E$.
- Given two $\mathbb{S}$-modules E, F, we let $E \circ F$ denote the $\mathbb{S}$-module such that

$$E \circ F(k) = \bigoplus_{k_1+\cdots+k_p=n} \Big(E(p) \otimes \Big(\mathrm{Ind}_{\mathbb{S}_{k_1}\times\cdots\times\mathbb{S}_{k_p}}^{\mathbb{S}_n} (F(k_1) \otimes \cdots \otimes F(k_p))\Big)\Big)^{\mathbb{S}_p},$$

  where the superscript notation denotes taking the invariants. One could make a definition of $E \tilde{\circ} F$ analogously, instead using the *co*induced representation and the $\mathbb{S}_p$ *co*invariants; this definition is more natural for definitions involving cooperads, but since we are in characteristic 0, there is a natural isomorphism $E \tilde{\circ} F \cong E \circ F$, so we make no distinction between the two.
- Given an Opens-colored $\mathbb{S}$-module E, we let $\mathcal{T}(E)$ denote the free (Opens-colored) operad generated by E. Given a non-negative integer k, an $\mathbb{R}$-linear basis for the space of k-ary operations in $\mathcal{T}(E)$ is given by the set of isomorphism classes of shuffle trees on E with k leaves. A shuffle tree on E is a rooted planar tree γ together with a labeling of the leaves of γ by the integers $\{1, \dots, k\}$, with a condition on this labeling which we now explain. By recursion, this labeling induces a labeling of all the internal edges of γ by integers, as follows: the output of vertex v is labeled by the minimum of all labels on inputs to v. A shuffle tree is a planar tree so labeled such that the labels on the inputs to a vertex are in increasing order from left-to-right. Finally, to take care of the colors and the module E, all the internal edges of γ have colors from Opens, and vertices of γ are labeled by elements of appropriate arity and color in E.
- Since we have fixed the spacetime manifold M and we almost exclusively consider colored operads whose colors are Opens, we reserve the right to use the term "operad" when "Opens-colored operad" is more precise.

1.4 Acknowledgments

The authors would like to thank Owen Gwilliam, Guillaume Laplante-Anfossi, and Stephan Stolz for helpful discussions and questions. E.R. would also like to thank Damien

Calaque, Víctor Carmona, and Thomas Willwacher for helpful discussions. N.I. was supported by the project HighAGT (ANR-20-CE40-0016), the project SHoCoS (ANR-22-CE40-0008), and the IdEx Université Paris Cité (ANR-18-IDEX-0001), and thanks the University of Notre Dame for its hospitality.

2 The operad encoding prefactorization algebras

In this section, we take for granted that the theory of Koszul duality for operads applies *mutatis mutandis* to the theory of colored operads. We introduce the necessary background concerning prefactorization algebras, and we describe a generators-and-relations presentation of the colored operad governing them. We fix once and for all a topological space M which will not generally appear in the notation.

In this section, we describe the operad Disj which encodes prefactorization algebras (see Section 2.2). Before we do so, we will first recall the definition of prefactorization algebras, the definition of operads, and how they are related.

2.1 Operads for factorization algebraists

There is no substitute for the excellent book of Loday–Vallette [LV12] on the subject of algebraic operads, and we use the results of that book heavily in this paper. Nevertheless, we will try to summarize the main points of the formalism for those who have not seen them before.

Operads, cooperads, bar and cobar constructions

A (symmetric) operad $\mathcal{O}$ is a collection of vector spaces (or chain complexes, or objects in a more general symmetric monoidal category) $\{\mathcal{O}(k)\}_{k=0}^{\infty}$ such that $\mathcal{O}(k)$ is a module for $\mathbb{S}_k$ (the symmetric group on k elements), together with a collection of maps

$$\circ : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)$$

which are associative and respect the $\mathbb{S}_n$-module structures in a natural way. For example, one may form operads Com, As, and Lie which encode commutative, associative, and Lie algebras, respectively. A number of similar definitions immediately present themselves. For example, by reversing the direction of the arrows, one obtains the notion of a cooperad. One may also impose a notion of unitality or counitality by privileging an element id of $\mathcal{O}(1)$ that acts as an identity (co)-operation, i.e., in the operad case, $\circ(\mu, \mathrm{id}, \ldots, \mathrm{id}) = \mu$. Finally, one may allow the output and each of the k inputs of an operation $\mu \in \mathcal{O}(k)$ to be labeled by elements of some fixed set of "colors" S, and to require the composite of operations to exist only when the output labels of the elements of $\mathcal{O}(n_1), \ldots \mathcal{O}(n_k)$ match the input labels of the element of $\mathcal{O}(k)$. In this way, one obtains the notion of a "colored operad." Henceforth, we will assume that the underlying symmetric monoidal category is Ch, and when we say "operad" (resp. "cooperad"), we will mean "unital (resp. counital) operad (resp. cooperad) in the symmetric monoidal category of chain complexes".

Given a cooperad $\mathcal{C}$ and an operad $\mathcal{P}$, the collection

$$\bigoplus_k \mathrm{Hom}_{\mathbb{S}_k}(\mathcal{C}(k), \mathcal{P}(k))$$

has a natural structure of a dg Lie algebra (the symbol Hom denotes here the internal hom in chain complexes). We thus obtain a natural bifunctor

$$\mathrm{Tw}(\_,\_) : \mathrm{CoOp} \times \mathrm{Op} \to \mathrm{Set}$$

which takes the Maurer-Cartan elements of this dg Lie algebra (Maurer-Cartan elements in this dg Lie algebra have the special name "twisting morphism"). It turns out that, once some small restrictions are placed on the categories of operads and cooperads under consideration, there is a left-right adjoint pair

$$\Omega : \mathrm{CoOp} \leftrightarrows \mathrm{Op} : B$$

which represent the bifunctor Tw, i.e., there are natural isomorphisms

$$\mathrm{Hom}_{\mathrm{Op}}(\Omega\mathcal{C}, \mathcal{P}) \cong \mathrm{Tw}(\mathcal{C}, \mathcal{P}) \cong \mathrm{Hom}_{\mathrm{CoOp}}(\mathcal{C}, B\mathcal{P}).$$

(The true domains and codomains of these functors are the categories of *conilpotent* cooperads and *augmented* operads.) The counit of the adjunction, $\Omega B\mathcal{P} \to \mathcal{P}$, induces a quasi-isomorphism of operads. The goal of the Koszul theory of operads is to provide a smaller resolution of $\mathcal{P}$ via an operad of the form $\Omega\mathcal{C}$, for some sub-cooperad $\mathcal{C} \hookrightarrow B\mathcal{P}$.

Koszul theory for associative algebras

But before we describe the Koszul theory of general operads, it is worthwhile to consider the special case of operads for which $\mathcal{O}(k) = 0$ unless $k = 1$. This is the case of (dg) associative, unital algebras.

As we mentioned parenthetically above, the functor B is defined only on the category of augmented operads; in the case of unital algebras, the operadic augmentation translates into an augmentation for associative algebras, i.e., we must consider algebras equipped with an algebra map $\epsilon : A \to \mathbb{R}$. So, given an augmented associative (possibly dg) algebra (A, μ, ϵ), we may form the bar-cobar resolution ΩBA, as for operads, and this algebra is generated by elements of the form

$$[a_1|\cdots|a_n] \in A^{\otimes n},$$

where each $a_i \in \ker(\epsilon)$ and n is any non-negative integer. Such an element has cohomological degree $1 - n + \sum_i |a_i|$, and the bar-cobar differential is defined by the equation

$$\begin{aligned} d[a_1|\cdots|a_n] = &\sum_{i=1}^{n-1} (-1)^{i+|a_1|+\cdots+|a_i|}[a_1|\cdots|\mu(a_i, a_{i+1})|\cdots|a_n] \\ &+ \sum_{i=1}^{n-1} (-1)^{|a_1|+\cdots+|a_i|-i+1}[a_1|\cdots|a_i] \cdot [a_{i+1}|\cdots|a_n], \end{aligned}$$

where the symbol $\cdot$ denotes the multiplication in the semi-free algebra ΩBA. So, whereas in the algebra A we had elements a, b whose product is $\mu(a,b)$, in the algebra ΩBA we have the elements $[a], [b]$ whose product is $[a] \cdot [b]$ and the equation $d[a|b] = \pm([\mu(a,b)] - [a] \cdot [b])$. In particular, if there is a relation of the form $\mu(a,b) = \mu(a',b')$ in A, this relation no longer holds on the nose in ΩBA; instead, one has

$$[a] \cdot [b] - [a'] \cdot [b'] = \pm d\left([a'|b'] - [a|b]\right).$$

Furthermore, the algebra ΩBA has homotopies between the homotopies $[a|b]$ and homotopies between those homotopies and so on.

This resolution has the benefit of being well-defined for any algebra A. Its drawback is that it is very large: even if A is finitely generated, ΩBA is not. Even more, in the bar-cobar resolution, the space of generators of weight one is not necessarily finite dimensional even when A is finitely generated (where the *weight* of $[a_1|\cdots|a_n]$ is n). To this end, it is desirable to look for smaller resolutions in case A is known to be described by a simple set of generators and relations. This is the goal of Koszul theory, which applies to algebras of the form $\mathcal{T}(V,R)$, where V is a vector space, $R \subseteq V^{\otimes 2} \oplus V \oplus \mathbb{R}$, and $\mathcal{T}(V,R)$ is the algebra generated by V subject only to the relations in R. The purpose of Koszul theory is to find a sub-coalgebra $A^{¡} \to BA$ such that the composite $\Omega A^{¡} \to \Omega BA \to A$ is still a quasi-isomorphism. In fact, for any algebra of the form $\mathcal{T}(V,R)$, there is a natural candidate for $A^{¡}$; one says that an algebra is *Koszul* if the map $\Omega A^{¡} \to A$ is a quasi-isomorphism.

The benefit of this construction is that if A is described by quadratic relations, then $A^{¡}$ is described by quadratic corelations, whereas BA is freely cogenerated by $A[1]$. This can dramatically reduce the number of generators of the semi-free resolution. For example, if $A = \mathcal{T}(x)$ is freely generated by an element x, then the bar-cobar resolution is freely generated by elements of the form

$$[x^{i_1}|\cdots|x^{i_n}];$$

by contrast, the resolution of A given by Koszul theory is just A again (since A was free to begin with, there was no need to provide it with a new resolution). In this case, $A^{¡} = \mathbb{R}\{1, \epsilon\}$ where $|\epsilon| = -1$ and $\Delta(\epsilon) = 1 \otimes \epsilon + \epsilon \otimes 1$. In general, $A^{¡}$ need not be finite-dimensional even if A is finitely-generated, so that the Koszul resolution of a finitely-generated algebra is not necessarily finitely-generated. But it will always be the case that the weight grading on BA descends to $A^{¡}$ and that the weight-one component of $A^{¡}$ is finite-dimensional if V is.

Mutatis mutandis, the preceding discussion applies equally well to more general operads. Moreover, many of the standard operads–including the associative, commutative, and Lie operads–are candidates for the application Koszul theory, since e.g., the associativity relation $\mu(\mu(a,b),c) = \mu(a,\mu(b,c))$ is quadratic in μ. Indeed, Koszul theory for these three operads produces the A_∞, C_∞, and L_∞ operads, respectively. In Section 2, we show that the operad Disj encoding prefactorization algebras on a fixed manifold M has a quadratic-linear generators-and-relations presentation (cf. Proposition 2.13); the main idea is that the operad is generated by the unary operations m_U^V for any inclusion $U \subset V$ and the binary operations $m_{U,V}^{U \sqcup V}$ for any disjoint pair of sets.

Relation of hoDisj algebras to (homotopy) commutative and associative algebras

Because this presentation can be a bit confusing at first glance, let's highlight what the structure maps ι_U^V and $\mu_{U,V}$ are in two common cases. In the first case, given a commutative algebra (C, μ), we can construct the prefactorization algebra $\mathcal{F}_C$ which assigns C to any open subset $U \subseteq M$ (M can be an arbitrary space). In this case, we have $\iota_U^V = \mathrm{id}_C$ and $\mu_{U,V} = \mu$, so the maps ι_U^V are "boring" and the maps $\mu_{U,V}$ are "interesting". In the second common case, let (A, μ, η) be an associative, unital algebra; one may form the factorization algebra $\mathcal{F}_A$ on $\mathbb{R}$ which assigns $\otimes_{\pi_0(U)} A$ to any open subset U of $\mathbb{R}$. In this case, the maps ι_U^V are "interesting" and the maps $\mu_{U,V}$ are "boring". More precisely, the map $\mu_{U,V}$ is given by the natural associator isomorphism

$$\left(\bigotimes_{\pi_0(U)} A\right) \otimes \left(\bigotimes_{\pi_0(V)} A\right) \to \bigotimes_{\pi_0(U \sqcup V)} A,$$

while the map ι_U^V is determined by the structures μ and η on A, e.g.,

$$\iota_\emptyset^{(0,1)} = \eta, \quad \iota_{(-1,0)\sqcup(0,1)}^{(-1,1)} = \mu.$$

In the previous paragraph, the assignment $C \mapsto \mathcal{F}_C$ gives a functor $\mathrm{Com}\text{-}\mathrm{Alg} \to \mathrm{Disj}\text{-}\mathrm{Alg}$; and it turns out that this assignment is induced via pullback from a map of operads $\mathrm{Disj} \to \mathrm{Com}$. There is also a map of operads the other way, which at the level of algebras sends a prefactorization algebra $\mathcal{F}$ to its value $\mathcal{F}(\emptyset)$ on the empty set. (It turns out that the two functors are adjoints, with the first functor the left adjoint.) These relationships will extend to the resolutions: namely, let hoDisj denote the resolution $\Omega\,\mathrm{Disj}^{\mathrm{i}}$ discussed above. Similarly, let C_∞ denote the resolution of Com obtained via Koszul theory. Then, there is a pair of functors

$$C_\infty\text{-}\mathrm{Alg} \leftrightarrows \mathrm{hoDisj}_M\text{-}\mathrm{Alg}\,. \tag{2.1}$$

By contrast, the functor $A \mapsto \mathcal{F}_A$ which maps a unital, associative algebra to a prefactorization algebra on $\mathbb{R}$ does not arise from a map of (colored) operads $\mathrm{Disj} \to \mathrm{As}$. Indeed, the structure maps of the algebra A appear in the unary generating operations of $\mathcal{F}_A$, as opposed to the binary operations. This suggests that the existence of the functor $A \mapsto \mathcal{F}_A$ has more to do with the topology of open subsets of the real line than with general algebraic properties of the colored operad Disj. Consequently, though one may imagine the existence, by analogy with Equation (2.1), of a functor

$$uA_\infty\text{-}\mathrm{Alg} \to \mathrm{hoDisj}_{\mathbb{R}}\text{-}\mathrm{Alg}$$

(where uA_∞ is the Koszul resolution of the operad governing unital associative algebras, see [HM12]), such a functor does not immediately present itself using our methods. We leave it as an open question to determine what sort of category of homotopy associative algebras forms the natural domain of a functor like the one above.

2.2 The operad Disj: definitions and basic properties

Definition 2.1. The colored $\mathbb{R}$-linear operad Disj has as its colors the open subsets $U \subseteq M$. Given open subsets $U_1, \dots, U_k, V \subseteq M$, we set:

$$\mathrm{Disj}\begin{pmatrix} V \\ U_1, \dots, U_k \end{pmatrix} := \begin{cases} \mathbb{R}\{m^V_{U_1,\dots,U_k}\}, & \text{if } U_i \subseteq V \text{ and the } U_i \text{ are pairwise disjoint;} \\ \{0\}, & \text{else.} \end{cases}$$

The composition map

$$\mathrm{Disj}\left({}^{\;W}_{V_1,\dots,V_k}\right) \otimes \mathrm{Disj}\left({}^{\;V_1}_{U_{11},\dots,U_{1n_1}}\right) \otimes \cdots \otimes \mathrm{Disj}\left({}^{\;V_k}_{U_{k1},\dots,U_{kn_k}}\right) \to \mathrm{Disj}\left({}^{\;W}_{U_{11},\dots,U_{kn_k}}\right)$$

is zero when any of the factors in the domain or codomain is zero, and otherwise is the natural isomorphism $\mathbb{R}^{\otimes(k+1)} \to \mathbb{R}$. A permutation $\sigma \in \mathbb{S}_k$ sends the generator $m^V_{U_1,\dots,U_k}$ to $m^V_{U_{\sigma^{-1}(1)},\dots,U_{\sigma^{-1}(k)}}$.

Definition 2.2. A **prefactorization algebra** is an algebra over Disj.

Definition 2.3. We let Opens denote the poset of open subsets of M. This poset defines a category, i.e., a colored operad with only unary operations.

Note that Opens is a sub-operad of Disj. Occasionally, when we want to make explicit the underlying space M, we will also write Disj_M or Opens_M. A prefactorization algebra consists of

1. A cochain complex $\mathcal{F}(U)$ for every open subset $U \subseteq M$.
2. A map
$$m^V_{U_1,\dots,U_k} : \mathcal{F}(U_1) \otimes \cdots \mathcal{F}(U_k) \to \mathcal{F}(V)$$
for any collection $\{U_i\}$ of pairwise disjoint open subsets of V.

These data are required to satisfy the following relations:

Symmetry Given a permutation $\sigma \in \mathbb{S}_k$, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) & \xrightarrow{\sigma} & \mathcal{F}(U_{\sigma(1)}) \otimes \cdots \otimes \mathcal{F}(U_{\sigma(k)}) \\ & \searrow^{m^V_{U_1,\dots,U_k}} & \downarrow{\scriptstyle m^V_{U_{\sigma(1)},\dots,U_{\sigma(k)}}} \\ & & \mathcal{F}(V) \end{array},$$

where the horizontal arrow is induced from the symmetric monoidal structure on the category of cochain complexes (including the usual Koszul signs).

Associativity Given any pairwise disjoint collection $\{V_j\}_{j=1}^r$ of subsets of W and, for each j, a collection $\{U_{ji}\}_{i=1}^{k_j}$ of pairwise disjoint subsets of V_j, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(U_{11})\otimes\cdots\otimes\mathcal{F}(U_{rk_r}) & \xrightarrow{\bigotimes_{j=1}^{r} m^{V_j}_{U_{j1},\dots,U_{jk_j}}} & \mathcal{F}(V_1)\otimes\cdots\otimes\mathcal{F}(V_r) \\ & \searrow^{m^{W}_{U_{11},\dots,U_{rk_r}}} & \Big\downarrow m^{W}_{V_1,\dots,V_r} \\ & & \mathcal{F}(W) \end{array}$$

Remark 2.4*:* There are maps of colored operads

$$\mathrm{Disj}\to\mathrm{Com},\qquad \mathrm{Com}\to\mathrm{Disj}\,.$$

The first map covers the map of sets of labels $\{*\}\to$ Opens which "picks out" the empty set amongst the open subsets of M. The second map covers the unique map of labels Opens $\to\{*\}$. At the level of algebras, the first map of operads extracts from a prefactorization algebra $\mathcal{F}$ the commutative algebra $\mathcal{F}(\emptyset)$, while the second map of operads assigns to a commutative algebra A the prefactorization algebra $\mathcal{F}_A$ such that $\mathcal{F}_A(U)=A$ for all open subsets $U\subseteq M$. We will see that, on account of these comparisons (in particular the latter one), the operad Disj behaves similarly to the commutative operad. ◊

Now, we give a generators-and-relations presentation of Disj which will enable us to compute its cofibrant resolution using Koszul duality theory.

Definition 2.5. Given open sets $U\subset V$, we let $\iota_U^V := m_U^V$ denote the (unary) generator of $\mathrm{Disj}\binom{V}{U}$. Given two disjoint open sets U and V, we let $\mu_{U,V} := m_{U,V}^{U\sqcup V}$ denote the (binary) generator of $\mathrm{Disj}\binom{U\sqcup V}{U,V}$.

Lemma 2.6. The operad Disj is generated by the operations of the form ι_U^V and $\mu_{U,V}$.

Proof. This follows from the following equation, which holds for any pairwise disjoint open sets $U_1,\dots,U_k$ contained in an open set V:

$$m^V_{U_1,\dots,U_k} = m^V_{U_1\sqcup\cdots\sqcup U_k}\circ_1 m^{U_1\sqcup\cdots\sqcup U_k}_{U_1\sqcup\cdots\sqcup U_{k-1},U_k}\circ_1 m^{U_1\sqcup\cdots\sqcup U_{k-1}}_{U_1\sqcup\cdots\sqcup U_{k-2},U_{k-1}}\circ_1\cdots\circ_1 m^{U_1\sqcup U_2}_{U_1,U_2}. \tag{2.2}$$

□

Definition 2.7. Let E be the Opens-colored $\mathbb{S}$-module defined as follows:

$$E(1)=\bigoplus_{U\subsetneq V}\mathbb{R}\{\iota_U^V\},\qquad E(2)=\bigoplus_{U\cap V=\emptyset}\mathbb{R}\{\mu_{U,V}\},\qquad E(k)=0 \text{ for } k\notin\{1,2\}.$$

We may form the free Opens-colored operad $\mathcal{T}(E)$ on E. This free operad is weight-graded, by the number of generating operations (i.e., the number of vertices in the trees). We let $\mathcal{T}^{(k)}(E)$ denote the weight k part of $\mathcal{T}(E)$.

Definition 2.8. Fix an ordered triple (U, V, W) of open subsets of M. Following [LV12, Section 7.6.3], we obtain the following $\mathbb{R}$-linear generators for the subspace of $\mathcal{T}^{(2)}(E)$ corresponding to trees whose vertices are labeled by the binary generators $\mu_{\_,\_}$:

$$\begin{aligned}
\tau_{\mathrm{I}}(U, V, W) &:= \mu_{U \sqcup V, W} \circ_1 \mu_{U,V}, \\
\tau_{\mathrm{II}}(U, V, W) &:= \mu_{W \sqcup U, V} \circ_1 \mu_{U,W} \cdot (23), \\
\tau_{\mathrm{III}}(U, V, W) &:= \mu_{U, V \sqcup W} \circ_2 \mu_{V,W}.
\end{aligned}$$

Figure 1 depicts these operations.

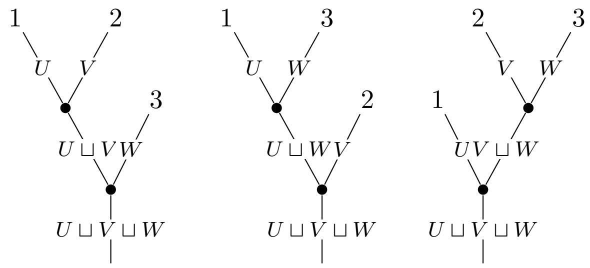


Figure 1: The generators $\tau_{\mathrm{I}}(U, V, W)$, $\tau_{\mathrm{II}}(U, V, W)$, and $\tau_{\mathrm{III}}(U, V, W)$ for the space of operations which have weight two and use only binary generators. All three are operations of color $\binom{U \sqcup V \sqcup W}{U,V,W}$.

Remark 2.9*:* To give a sense of how the symmetric group acts on these generators, note the relation:

$$\tau_{\mathrm{I}}(U, V, W) \cdot (123) = \tau_{\mathrm{II}}(U, W, V)$$

◊

Definition 2.10. Let $R := R_1 \oplus R_2 \oplus R_3$ denote the sub-$\mathbb{S}$-module of $\mathcal{T}^{(2)}(E) \oplus E$ defined by:

$$\begin{aligned}
R_1 &:= \bigoplus_{U \subset V \subset W} \mathbb{R}\left\{ \iota_V^W \circ \iota_U^V - \iota_U^W \right\}, \\
R_2 &:= \bigoplus_{\substack{(U,V,W) \\ U \cap V = \emptyset \\ U \cap W = \emptyset \\ V \cap W = \emptyset}} \mathbb{R}\left\{ \tau_{\mathrm{I}}(U, V, W) - \tau_{\mathrm{II}}(U, V, W), \tau_{\mathrm{II}}(U, V, W) - \tau_{\mathrm{III}}(U, V, W) \right\} \\
R_3 &:= \bigoplus_{\substack{U \subset V \\ V \cap W = \emptyset}} \mathbb{R}\{ \mu_{V,W} \circ_1 \iota_U^V - \iota_{U \sqcup W}^{V \sqcup W} \circ_1 \mu_{U,W} \} \\
&\quad \oplus \bigoplus_{\substack{W \subset V \\ V \cap U = \emptyset}} \mathbb{R}\{ \mu_{U,V} \circ_2 \iota_W^V - \iota_{U \sqcup W}^{U \sqcup V} \circ_1 \mu_{U,W} \}.
\end{aligned}$$

We have written R in this way to highlight a few major points. Note that the space of relations is split depending on the type of generators used (exclusively unary, exclusively binary, or mixed). The space R_1 consists of unary operations, R_2 of ternary operations, and R_3 of binary operations. Moreover,

$$R_2, R_3 \subseteq \mathcal{T}^{(2)}(E),$$

i.e., only R_1 presents quadratic-linear relations.

Proposition 2.11. The operad $\mathcal{T}(E(1), R_1)$ (generated by the ι_U^V, subject to the relations in R_1) is isomorphic to Opens, the operad encoding precosheaves on M (Definition 2.3).

Proof. A generic element of $\mathcal{T}(E(1), R_1)$ is of the form

$$\iota_{U_k}^{U_{k-1}} \circ \cdots \circ \iota_{U_1}^{U_0},$$

with $U_0 \subset \cdots \subset U_k$. Applying repeatedly the relations in R_1, we see that this element is equal to $\iota_{U_k}^{U_0}$. The morphism of operads $\mathcal{T}(E(1), R_1) \to \mathrm{Opens}$ is thus bijective. ☐

Definition 2.12. Let Tens be the operad $\mathcal{T}(E(2), R_2)$, i.e., the colored binary operad generate by elements of the form $\mu_{U,V}$ subject to the relations in R_2. We will call algebras over Tens **tensor systems**.

This colored operad resembles a colored version of the commutative operad. The relation is made precise in Section 5.2.

The space R_3 encodes the relations between the binary generators and the unary ones. Given any composable unary operation ι and binary operation μ, the relations in R_3 tell us how to rewrite the composites $\mu \circ_1 \iota$ and $\mu \circ_2 \iota$ in terms of a composition where the binary operation is performed first. We we will see (Lemma 3.3) that this gives a rewriting rule: namely, one can write

$$\mathcal{T}(E, R) \cong \mathrm{Opens} \circ \mathrm{Tens}\,.$$

We will see in the next proposition that $\mathcal{T}(E, R) \cong \mathrm{Disj}$ (Proposition 2.13). Hence, we can conclude that Disj can be written as a composition product of the operads Opens and Tens.

Having established the notation, we may prove:

Proposition 2.13. There is an isomorphism $\Phi : \mathcal{T}(E, R) \to \mathrm{Disj}$ of Opens-colored operads.

Proof. We have already remarked upon a few parts of the proof. Define

$$\Phi(\mu_{U,V}) := m_U^V, \qquad\qquad \Phi(\iota_U^V) := m_{U,V}^{U \sqcup V}\,.$$

By the universal property of the free operad, these equations suffice to define a map $\mathcal{T}(E) \to \mathrm{Disj}$, which we also denote using the letter Φ. It is immediate that Φ vanishes on R and thus descends to a map $\mathcal{T}(E, R) \to \mathrm{Disj}$. Equation (2.2) shows that Φ is arity-wise surjective.

To show that Φ is arity-wise injective, consider an operation μ of color

$$\binom{V}{U_1, \ldots, U_k}$$

in $\mathcal{T}(E)$. The operation μ can be represented (non-uniquely) by a planar rooted tree with bivalent and trivalent vertices, together with a bijection from the set of leaves to $\{1, \ldots, k\}$. Each edge, including those incident on the leaves and root, has a label from the poset Opens. Furthermore, if the incoming edges of a trivalent vertex have colors U and V, then the output has to have color $U \sqcup V$. We will show that, modulo (R), μ is of the form appearing in the right-hand-side of Equation (2.2). Let us observe first that any ι_U^V operation which is precomposed with a $\mu_{U,V}$ operation can be moved to the output of the $\mu_{U,V}$ operation using R_3 relations. Hence, modulo (R), μ is equivalent to a rooted planar binary tree with a "ladder" attached to the root. Necessarily, the first input color on the ladder is $U_1 \sqcup \cdots \sqcup U_k$, and the final input color on the ladder is V. Next, we use R_1 to replace this ladder of ι_U^V operations by $\iota_{U_1 \sqcup \cdots \sqcup U_k}^V$. Finally, we use the associativity relation in R_2 to turn the tree into a "left comb." When read from left to right, the labels on the leaves may not be in order. Here again, we may use R_2. Figure 2 shows how this works in an example.

In other words, in arity

$$\binom{V}{U_1, \ldots, U_k},$$

the operad $\mathcal{T}(E, R)$ is a vector space of dimension at most 1, and Φ is a surjective map from this space to a line. Hence, $\mathcal{T}(E, R)$ must be a line in this arity, and Φ must be an isomorphism. □

Proposition 2.13 gives a quadratic-linear, generators-and-relations presentation of the operad Disj. In light of the proposition, we will cease to make a distinction between $\mathcal{T}(E, R)$ and Disj. We would like to apply Koszul duality theory to the operad Disj. To do this, we need to check first that the colored operad Disj satisfies the properties (ql1) and (ql2) detailed in [LV12, Section 7.8]. These conditions pertain to the minimality of the set of generators and the maximality of the relations; we must check them because our presentation is quadratic-linear. Next, we need to check that the operad Disj is Koszul. We undertake the first task here, and the second task below.

Lemma 2.14. The operad $\mathrm{Disj} = \mathcal{T}(E, R)$ satisfies the conditions for a quadratic-linear operad from [LV12, Section 7.8]:

(ql1) $R \cap E = \{0\}$;

(ql2) $\{R \circ_{(1)} E + E \circ_{(1)} R\} \cap \mathcal{T}^{(2)}(E) \subseteq R \cap \mathcal{T}^{(2)}(E)$.

Remark 2.15*:* The condition (ql1) is a minimality condition for the generators: if the relations set some generators to zero, then those generators should be excluded to satisfy (ql1). The condition (ql2) is a maximality condition for the set of relations. Indeed, $R \circ_{(1)} E + E \circ_{(1)} R$ consists of (cubic-quadratic) relations which hold in $\mathcal{T}(E, R)$ as a

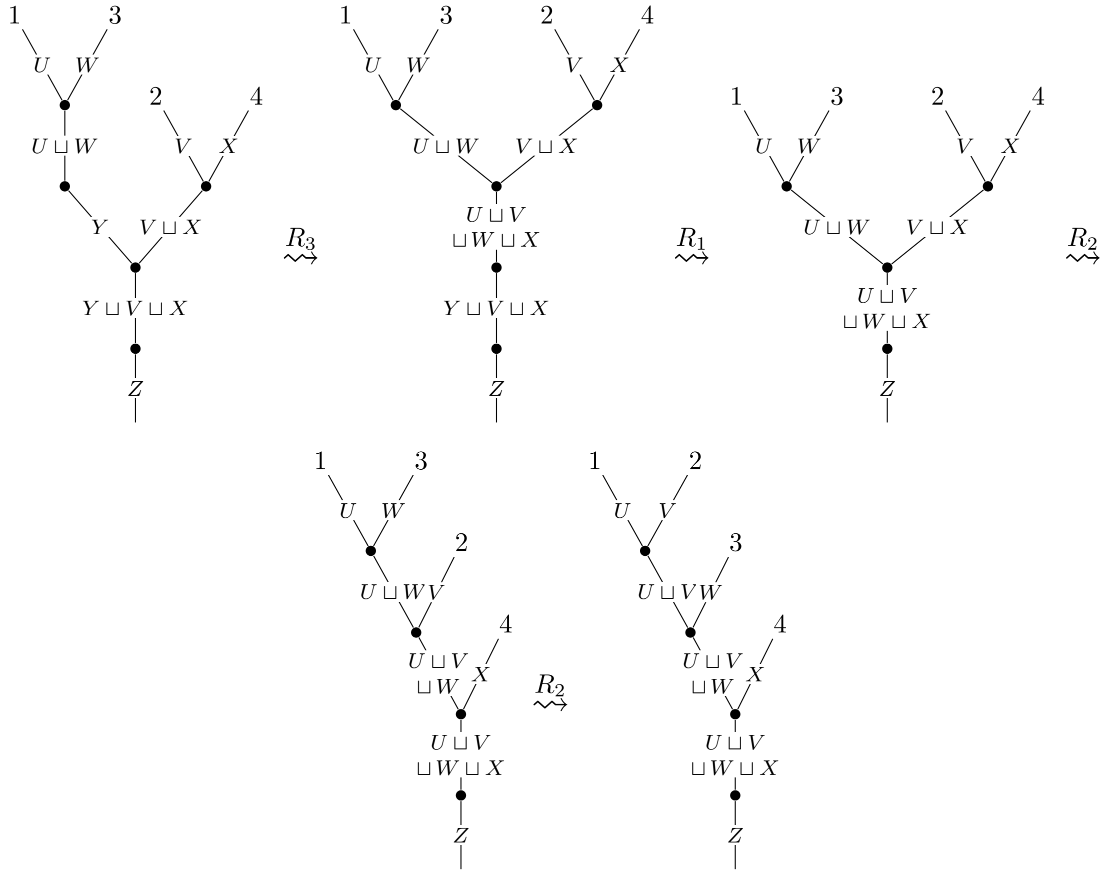


Figure 2: Applying the relations to reduce a tree to a standard form.

consequence of the relations R. Since R has quadratic-linear relations, some of these relations could be purely quadratic as cubic terms could cancel. However, the condition (ql2) ensures that these quadratic relations are already in R. $\Diamond$

Proof. The property (ql1) is immediate.

For (ql2), let us describe the plan of attack. We have described a basis for E and a basis for R. This allows one to give a relatively straightforward description of the set

$$S \coloneqq \{E \circ_{(1)} R + R \circ_{(1)} E\}.$$

In general, this is a subset of $\mathcal{T}^{(3)}(E) \oplus \mathcal{T}^{(2)}(E)$. The intersection of $S \cap \mathcal{T}^{(2)}(E)$ can be found by identifying all cubic terms arising in S and then studying how to cancel them with each other. After obtaining a cancellation in this way, we note any quadratic terms which remain and check whether or not these belong to R.

We can group the possible cubic terms by the nature of their vertices:

Three extension maps ι_U^V The only possibility for such terms is by composing R_1 with a ι_U^V map. In this way, we obtain—for any sequence $U \subseteq V \subseteq W \subseteq X$—the relations

$$\iota_W^X \circ \iota_V^W \circ \iota_U^V - \iota_V^X \circ \iota_U^V, \quad \iota_W^X \circ \iota_V^W \circ \iota_U^V - \iota_W^X \circ \iota_U^W.$$

The only way to cancel the cubic terms above is to subtract the two terms; in this way, we obtain the relation

$$\iota_V^X \circ \iota_U^V - \iota_W^X \circ \iota_U^W = \left(\iota_V^X \circ \iota_U^V - \iota_U^X\right) - \left(\iota_W^X \circ \iota_U^W - \iota_U^X\right);$$

this term manifestly lies in R_1.

Three binary operations $\mu_{U,V}$ These can only be obtained from composing R_2 with a binary generator. Since R_2 is purely quadratic, such terms all lie in $\mathcal{T}^{(3)}(E)$, with no terms in $\mathcal{T}^{(2)}(E)$. Hence,

$$\{R_2 \circ_{(1)} \mathbb{R}\{\mu_{U,V}\} + \mathbb{R}\{\mu_{U,V}\} \circ_{(1)} R_2\} \cap \mathcal{T}^{(2)}(E) = 0.$$

Two binary operations $\mu_{U,V}$ These can be generated by a relation in R_3 composed with a binary generator, or a relation in R_2 composed with a unary generator. Since R_2 and R_3 are purely quadratic, this case is dealt with as in the case immediately preceding.

Two unary operations ι_U^V This is the only case that presents difficulties. Such terms are created by composing an R_1 relation with a binary generator or an R_3 relation with a unary generator. Let us give a full accounting of the composites that may appear in this way, fixing the two input sets U, V and the output set W. We obtain the following spanning set for the subspace S' of S whose cubic terms involve exactly two unary operations:

$$\iota^{W}_{W'} \circ \iota^{W'}_{U\sqcup V} \circ \mu_{U,V} - \iota^{W}_{U\sqcup V} \circ \mu_{U,V} \tag{e_1}$$

$$\mu_{U'',V} \circ_1 (\iota^{U''}_{U'} \circ \iota^{U'}_{U}) - \mu_{U'',V} \circ_1 \iota^{U''}_{U}, \qquad (W = U'' \sqcup V) \tag{e_2}$$

$$\mu_{U,V''} \circ_2 (\iota^{V''}_{V'} \circ \iota^{V'}_{V}) - \mu_{U,V''} \circ_2 \iota^{V''}_{V}, \qquad (W = U \sqcup V'') \tag{e_3}$$

$$(\mu_{U',V'} \circ_1 \iota^{U'}_{U}) \circ_2 \iota^{V'}_{V} - \iota^{U'\sqcup V'}_{U\sqcup V'} \circ (\mu_{U,V'} \circ_2 \iota^{V'}_{V}), \qquad (W = U' \sqcup V') \tag{e_4}$$

$$\mu_{U'',V} \circ_1 (\iota^{U''}_{U'} \circ \iota^{U'}_{U}) - \iota^{U''\sqcup V}_{U'\sqcup V} \circ (\mu_{U',V} \circ_1 \iota^{U'}_{U}), \qquad (W = U'' \sqcup V) \tag{e_5}$$

$$\iota^{W}_{U'\sqcup V} \circ (\mu_{U',V} \circ_1 \iota^{U'}_{U}) - \iota^{W}_{U'\sqcup V} \circ \iota^{U'\sqcup V}_{U\sqcup V} \circ \mu_{U,V} \tag{e_6}$$

$$(\mu_{U',V'} \circ_1 \iota^{U'}_{U}) \circ_2 \iota^{V'}_{V} - \iota^{U'\sqcup V'}_{U\sqcup V'} \circ (\mu_{U',V} \circ_1 \iota^{U'}_{U}), \qquad (W = U' \sqcup V') \tag{e_7}$$

$$\mu_{U,V''} \circ_2 (\iota^{V''}_{V'} \circ \iota^{V'}_{V}) - \iota^{U\sqcup V''}_{U\sqcup V'} \circ (\mu_{U,V'} \circ_2 \iota^{V'}_{V}), \qquad (W = U \sqcup V'') \tag{e_8}$$

$$\iota^{W}_{U\sqcup V'} \circ (\mu_{U,V'} \circ_2 \iota^{V'}_{V}) - \iota^{W}_{U\sqcup V'} \circ \iota^{U\sqcup V'}_{U\sqcup V} \circ \mu_{U,V} \tag{e_9}$$

Here, we require the strict inclusions $U \subsetneq U' \subsetneq U''$, $V \subsetneq V' \subsetneq V''$, and $W' \subsetneq W$, and a set of generators for S' is obtained by letting $U', V', \dots$ vary over all such. Some of the generators only appear if W is of a specific form, as noted in the enumeration. We will see that, without loss of generality in the present proof, we will be able to assume that W satisfies all conditions simultaneously. Consider a general element $e \in S'$. We write

$$e = \sum_{i=1}^{9} c_i e_i,$$

where $c_2 = 0$ unless $W = U'' \sqcup V$, and similarly for c_3, c_4, c_5, c_7, and c_8. Suppose further that $e \in S' \cap \mathcal{T}^{(2)}(E)$. This imposes equations on the c_i by setting to zero the coefficients of any given cubic composition of generators appearing in the e_i. Furthermore, we may, without loss of generality, make two "fine-tuning" assumptions: first, that W is of the necessary form as specified in the definition of each e_i (e.g. $W = U'' \sqcup V$) and second, that W' is both of the form $U \sqcup V'$ and of the form $U' \sqcup V$. Indeed, to pass to the "non-fine-tuned" situation one replaces a single equation of the form

$$f_1(c_1, \dots, c_9) + f_2(c_1, \dots, c_9) = 0,$$

with the set of two equations

$$f_1(c_1, \dots, c_9) = 0, \quad f_2(c_1, \dots, c_9) = 0$$

(and repeats this process if necessary). In other words, we find that the "non-fine-tuned" solutions are a subspace of the "fine-tuned" solutions. Proceeding with the fine-tuned case, we obtain the six equations

$$\begin{aligned} c_1 - c_6 - c_9 &= 0 \\ c_2 + c_5 &= 0 \\ c_3 + c_8 &= 0 \\ c_4 + c_7 &= 0 \\ -c_5 + c_6 - c_7 &= 0 \\ -c_4 - c_8 + c_9 &= 0. \end{aligned}$$

Each of these equations represents one of the six planar trees with two unary vertices and one binary vertex, e.g., the first equation corresponds to the diagram in Figure 3a. Putting the coefficients $c_1, \ldots, c_9$ into a column vector, one obtains the following basis for the space of solutions to these equations:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

One can verify directly that in all three cases, the corresponding sum $\sum_i c_i e_i$ is indeed quadratic in the generating operations. One therefore finds that the space $S' \cap \mathcal{T}^{(2)}(E)$ is the space of all linear combinations

$$-c_1(\iota^W_{U\sqcup V} \circ \mu_{U,V}) - c_2(\mu_{U'',V} \circ_1 \iota^{U''}_U) - c_3(\mu_{U,V''} \circ_2 \iota^{V''}_V) \tag{2.3}$$

where $c_1 + c_2 + c_3 = 0$. In the case that we have been considering, namely that all three summands in Equation (2.3) have the same output color, one verifies directly the element in that equation belongs to R_3. Similarly, if $W \neq U'' \sqcup V$ but $W = U \sqcup V''$, then we have $c_2 = 0$ and the element in Equation (2.3) again belongs to R_3. The remaining cases (setting first $c_3 = 0, c_2 \neq 0$ and second $c_3 = c_2 = 0$) are dealt with in a similar manner. □

(a) $c_1 - c_6 - c_9 = 0$

(b) $-c_4 - c_8 + c_9 = 0$

Figure 3: The cubic trees corresponding to two of the equations for the c_i.

3 Proof of the Koszul property

Lemma 3.1. The quadratic colored operad $q\,\mathrm{Opens}$ is Koszul.

Proof. The colored operad $q\,\mathrm{Opens}$ is a colored version of the dual numbers algebra. Its Koszul dual cooperad is the cofree colored cooperad on generators τ_U^V of degree -1 for each strict inclusion $U \subsetneq V$. The proof of the Koszulity of the dual numbers algebra applies equally well here. ☐

Lemma 3.2. The quadratic colored operad Tens is Koszul.

Proof. We need to show that the Koszul complex $\mathrm{Tens}^{¡} \circ_\kappa \mathrm{Tens}$ is acyclic.

The operad Tens is generated by the binary operations $\mu_{U,V}$ subject to the relations R_2. For this reason, it resembles a colored version of the commutative operad. Our strategy is therefore to reduce the case under consideration to the proof of the Koszulity of the commutative operad. To this end, note that the space

$$\mathrm{Tens}\begin{pmatrix} V \\ U_1, \dots, U_k \end{pmatrix}$$

is one-dimensional if the U_i are pairwise disjoint and $V = U_1 \sqcup \dots \sqcup U_k$; the space is 0 otherwise. In other words, if one fixes the input colors of an operation, the output color is determined by the inputs. When it is non-zero, the corresponding space of operations agrees with the one for the commutative operad. We will use variations of this basic observation repeatedly.

Now, we wish to understand the Koszul dual cooperad $\mathrm{Tens}^{¡}$. Let us first consider the cofree cooperad $\mathcal{T}^c(E(2)[1])$ on the binary generators $\mu_{U,V}$. The Koszul dual cooperad $\mathrm{Tens}^{¡}$ is a sub-cooperad of $\mathcal{T}^c(E(2)[1])$ which we will consider in a moment. Let us fix a collection $U_1, \dots, U_k$ of input colors. A straightforward induction shows that

$$\mathcal{T}^c(E(2)[1])\begin{pmatrix} V \\ U_1, \dots, U_k \end{pmatrix}$$

is the zero vector space unless the U_i are pairwise disjoint and $V = U_1 \sqcup \dots \sqcup U_k$. In the latter case, the vector space has a basis consisting of binary shuffle trees (the input colors determine the operations at the vertices and the colors on the edges and root of the tree). In other words,

$$\mathcal{T}^c(E(2)[1])\begin{pmatrix} V \\ U_1, \dots, U_k \end{pmatrix} = \mathcal{T}^c(F[1])(k),$$

where F is the trivial $\mathbb{S}_2$-module concentrated in arity 2. To lie in $\mathrm{Tens}^{¡}$, a linear combination of trees needs to have relations from R_2 on all pairs of adjacent vertices. Fixing pairwise disjoint input colors U, V, W, the space of relators

$$R_2\begin{pmatrix} U \sqcup V \sqcup W \\ U, V, W \end{pmatrix}$$

is naturally identified with the space of relations defining the single-colored operad governing commutative algebras. Hence, we obtain

$$\mathrm{Tens}^{\mathsf{i}}\begin{pmatrix} V \\ U_1,\dots,U_k \end{pmatrix} = \begin{cases} \mathrm{Com}^{\mathsf{i}}(k) = \mathrm{Lie}^c\{1\}, & V = U_1 \sqcup \cdots \sqcup U_k; \\ 0 & \text{else.} \end{cases}$$

Moreover, given $U_{11},\dots,U_{1p_1},\dots,U_{kp_k}$ pairwise disjoint, set $W_i = U_{i1} \sqcup \cdots \sqcup U_{ip_i}$ and $V = W_1 \sqcup \cdots \sqcup W_k$. The only non-zero cocomposition map

$$\mathrm{Tens}^{\mathsf{i}}\begin{pmatrix} V \\ U_{11},U_{12},\dots,U_{kp_k} \end{pmatrix} \to \mathrm{Tens}^{\mathsf{i}}\begin{pmatrix} V \\ W_1,\dots,W_k \end{pmatrix} \otimes \bigotimes_{i=1}^{k} \mathrm{Tens}^{\mathsf{i}}\begin{pmatrix} W_i \\ U_{i1},\dots,U_{ip_i} \end{pmatrix}$$

is given by the cocomposition in $\mathrm{Lie}^c\{1\}$.

Finally, let us consider the Koszul complex $\mathrm{Tens}^{\mathsf{i}} \circ_\kappa \mathrm{Tens}$. Let us fix a set of input colors $U_1,\dots,U_k$. Unless the U_i are pairwise disjoint and $V = U_1 \sqcup \cdots \sqcup U_k$, we have

$$\left(\mathrm{Tens}^{\mathsf{i}} \circ_\kappa \mathrm{Tens}\right)\begin{pmatrix} V \\ U_1,\dots,U_k \end{pmatrix} = 0.$$

In the case that the Koszul complex is non-zero, it is precisely the Koszul complex for Com in arity k, hence is acyclic. ☐

Lemma 3.3. Consider the quadratic operad $\mathcal{T}(E,qR)$ associated to the quadratic-linear operad $\mathrm{Disj} \cong \mathcal{T}(E,R)$. The natural map

$$\Psi : q\,\mathrm{Opens} \circ \mathrm{Tens} \to \mathcal{T}(E,qR)$$

of colored $\mathbb{S}$-modules is an isomorphism.

Proof. Note the following: the colored operad $\mathcal{T}(E,qR)$ is obtained via a rewriting rule

$$\lambda : \mathrm{Tens} \circ_{(1)} q\,\mathrm{Opens} \to q\,\mathrm{Opens} \circ_{(1)} \mathrm{Tens}\,.$$

Indeed, the relation R_3 tells us precisely how to turn any composite in $\mathrm{Tens} \circ_{(1)} q\,\mathrm{Opens}$ into a composite in $q\,\mathrm{Opens} \circ_{(1)} \mathrm{Tens}$. Hence, we may write

$$\mathcal{T}(E,qR) = q\,\mathrm{Opens} \vee_\lambda \mathrm{Tens},$$

and the rewriting rule allows us to conclude that the map Ψ is an arity-wise surjection.

Let $q\,\mathrm{Disj}$ denote the colored operad whose underlying colored $\mathbb{S}$-module is the same as Disj, but such that we may only compose $\mu^V_{U_1,\dots,U_k}$ with $\mu^W_{V_1,\dots,V_{k'}}$ non-trivially, where $V = V_i$, if either $V = U_1 \sqcup \cdots \sqcup U_k$ or $W = V_1 \sqcup \cdots V_{i-1} \sqcup \cdots \sqcup V_{k'}$. It is manifest that, as a colored $\mathbb{S}$-module,

$$q\,\mathrm{Disj} \cong q\,\mathrm{Opens} \circ \mathrm{Tens}\,.$$

Moreover, there is a natural map of operads

$$\Psi' : \mathcal{T}(E,qR) \to q\,\mathrm{Disj}$$

constructed exactly as in the proof of Proposition 2.13, and this map is an isomorphism for the same reasons as in the proof there. The composite map $\Psi' \circ \Psi$ is readily seen to be an isomorphism of $\mathbb{S}$-modules; hence, Ψ is a monomorphism, which completes the proof. □

Theorem 3.4. The operad Disj is Koszul.

Proof. Since $\mathrm{Disj} \cong \mathcal{T}(E, R)$ is a quadratic-linear operad, Koszulity of Disj is defined to be Koszulity of the associated quadratic operad $\mathcal{T}(E, qR)$. By Lemma 3.3, $\mathcal{T}(E, qR)$ can be written as a composition product via a rewriting rule. By the Diamond Lemma, Koszulity of $\mathcal{T}(E, qR)$ follows from the Koszulity of the factors, which is the content of Lemmas 3.1 and 3.2. □

We can use the preceding arguments to describe the Koszul dual $\mathcal{T}(E, R)^{\text{¡}}$. Before we do this, however, let us establish a bit of notation. Note that, because we have applied the diamond lemma to the operad $q\mathcal{T}(E, R)$, by [Bel18, Theorem B.3], we may write $q\mathcal{T}(E, R)^{\text{¡}} \cong \mathrm{Tens}^{\text{¡}} \circ q\,\mathrm{Opens}^{\text{¡}}$. Given disjoint subsets $U_{1s_1}, \dots, U_{ks_k}$, we let $\mu^{\text{¡}}_{U_{1s_1},\dots,U_{ks_k}}$ denote the image of the k-ary operation under the map

$$\mathrm{As}^{\text{¡}}(k) \to \mathrm{Com}^{\text{¡}}(k) \cong \mathrm{Tens}^{\text{¡}} \begin{pmatrix} U_{1s_1} \sqcup \cdots \sqcup U_{ks_k} \\ U_{1s_1}, \dots, U_{ks_k} \end{pmatrix}.$$

(Note that the while the $\mu^{\text{¡}}_{U_1,\dots,U_k}$ generate the spaces of k-ary cooperations as $\mathbb{S}$-modules, there are relations between these operations.) Let $(\iota^{\text{¡}})^V_U$ denote the cogenerator in $\mathcal{T}^c(sE, s^2R) = \mathcal{T}(E, R)^{\text{¡}}$ corresponding to ι^V_U. For each $1 \le i \le k$, given a chain $\mathcal{U}_i = (U_{i1} \subsetneq U_{i2} \subsetneq \cdots \subsetneq U_{is_i})$ of inclusions, we let:

$$\iota^{\text{¡}}_{\mathcal{U}_i} := (\iota^{\text{¡}})^{U_{is_i}}_{U_{i(s_i-1)}} \underline{\circ} \cdots \underline{\circ} (\iota^{\text{¡}})^{U_{i2}}_{U_{i1}},$$

where the notation $\underline{\circ}$ denotes the formal "composite" in the cooperad $(q\,\mathrm{Opens})^{\text{¡}}$ (if $s_i = 1$, we understand $\iota^{\text{¡}}_{\mathcal{U}_i}$ to mean the identity). Finally, we use $\mathcal{U}$ to denote the full (ordered) collection $(\mathcal{U}_1, \dots, \mathcal{U}_k)$, and define

$$\mu^{\text{¡}}_{\mathcal{U}} = (\mu^{\text{¡}}_{U_{1s_1},\dots,U_{ks_k}}; \iota^{\text{¡}}_{\mathcal{U}_1}, \dots, \iota^{\text{¡}}_{\mathcal{U}_k});$$

given a $(s_1 - 1, \dots, s_k - k)$ shuffle σ, we let $\sigma \cdot \mathcal{U}$ denote the unique chain of inclusions starting with $U_{11} \sqcup \cdots \sqcup U_{k1}$ and ending with $U_{1s_1} \sqcup \cdots \sqcup U_{ks_k}$ corresponding to σ (see below for an example).

Lemma 3.5. The cooperadic (co)distributive law defining the cooperadic structure on

$$(q\mathcal{T}(E, R))^{\text{¡}} \cong \mathrm{Tens}^{\text{¡}} \circ \mathrm{Opens}^{\text{¡}}$$

is given by the map

$$\begin{aligned} &\Lambda^c : \mathrm{Tens}^{\text{¡}} \circ \mathrm{Opens}^{\text{¡}} \to \mathrm{Opens}^{\text{¡}} \circ \mathrm{Tens}^{\text{¡}} \\ &\Lambda^c(\mu^{\text{¡}}_{\mathcal{U}}) = \sum_{\sigma \in \mathrm{Sh}(s_1-1,\dots,s_k-1)} (-1)^{(k-1)\sum(s_i-1)} \operatorname{sgn}(\sigma) \iota^{\text{¡}}_{\sigma\cdot\mathcal{U}} \,\underline{\circ}\, \mu^{\text{¡}}_{U_{11},\dots,U_{k1}}. \end{aligned} \tag{3.1}$$

The differential in the Koszul dual is given by

$$
\begin{aligned}
d\mu^{\text{i}}_{\mathcal{U}} &= \sum_{i=1}^{k} (-1)^{(k-1)+(\sum_{j=1}^{i-1}(s_j-1))} (\mu^{\text{i}}_{U_{1s_1},\ldots,U_{ks_k}}; \iota^{\text{i}}_{\mathcal{U}_1}, \ldots, d\iota^{\text{i}}_{\mathcal{U}_i}, \ldots, \iota^{\text{i}}_{\mathcal{U}_k}) \\
&:= \sum_{i=1}^{k} (-1)^{(k-1)+(\sum_{j=1}^{i-1}(s_j-1))} \mu^{\text{i}}_{d_i\mathcal{U}}
\end{aligned}
\tag{3.2}
$$

here,

$$
d\iota^{\text{i}}_{\mathcal{U}_i} = \sum_{j=1}^{s_i-2} (-1)^{j-1} \iota^{\text{i}}_{\mathcal{U}_i,\hat{j}},
$$

where $\iota^{\text{i}}_{\mathcal{U}_i,\hat{j}}$ is the result of replacing $(\iota^{\text{i}})^{U_{i(j+2)}}_{U_{i(j+1)}} \underline{\circ} (\iota^{\text{i}})^{U_{i(j+1)}}_{U_{ij}}$ with $(\iota^{\text{i}})^{U_{i(j+2)}}_{U_{ij}}$ in $\iota^{\text{i}}_{\mathcal{U}_i}$.

Remark 3.6*:* To clarify the meaning of the symbol $\iota^{\text{i}}_{\sigma\cdot\mathcal{U}}$, let us describe it in a simple example: let $k = 2$, $s_1 = 3$, $s_2 = 2$, so that $\mathcal{U}$ consists of a chain $U_{11} \subsetneq U_{12} \subsetneq U_{13}$ and a disjoint chain $U_{21} \subsetneq U_{22}$. Let $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 2$. Then,

$$
\iota^{\text{i}}_{\sigma\cdot\mathcal{U}} = (\iota^{\text{i}})^{U_{13}\sqcup U_{22}}_{U_{12}\sqcup U_{22}} \underline{\circ} (\iota^{\text{i}})^{U_{12}\sqcup U_{22}}_{U_{12}\sqcup U_{21}} \underline{\circ} (\iota^{\text{i}})^{U_{12}\sqcup U_{21}}_{U_{11}\sqcup U_{21}}. \qquad \lozenge
$$

Proof. The claim about the differential follows directly from the definitions. The most subtle thing in the proof thereof is the Koszul sign rule.

We have already established the Koszulity of $q\,\text{Disj}$ using the diamond lemma. It follows from [Bel18, Proposition B.2] that the Koszul dual $q\,\text{Disj}^{\text{i}}$ can be described by a codistributive law, which is itself induced from the rewriting rule λ used in the proof of Lemma 3.3 It remains to make explicit the consequences of this codistributive law. To this end, we make heavy reference to the proof of [Bel18, Lemma 2.2], where a similar proof is undertaken for an operad generated by:

- a binary operation of degree +1,
- a unary operation of degree 0

subject to the relations

- the binary operation satisfies the Jacobi relation
- the unary operation squares to 0
- the binary and unary operations satisfy the same rewriting rule as in $q\,\text{Disj}$.

The present case differs from the case studied in [Bel18] in the following three ways:

1. there are additional open subsets of M labeling all operations
2. the binary operators form an analogue of the commutative operad rather than the Lie operad, and
3. the binary operations have degree 0 rather than degree +1.

Item 1 presents only the challenge of notational complexity; item 2 is immaterial; and item 3 is responsible for the sign $(-1)^{(k-1)\sum(s_i-1)}$ appearing in Equation (3.1). The proof of Equation (3.1) is, as in [Bel18], by a double induction on the number of binary cogenerators and the number of unary cogenerators. Since the proof is similar to that case, we refer the reader thither for all the details; here, we will comment on the necessary modifications in the present case and on a few points which are omitted in the proof in [Bel18] but which we found enlightening to understand. The first difference lies in establishing the base cases for the induction. We need to establish the base case that $k = 2$, $s_1 + s_2 = 2$. This corresponds to tracing

$$(\mu^{\mathrm{i}}_{U,V}; (\iota^{\mathrm{i}})^{U}_{U'}, \mathrm{id})$$

(and the composition in the other factor) through the codistributive law. The codistributive law is the composite

$$\mathrm{Tens}^{\mathrm{i}} \circ (q\,\mathrm{Opens})^{\mathrm{i}} \to \mathcal{T}^c(E[1], R[2]) \to (q\,\mathrm{Opens})^{\mathrm{i}} \circ \mathrm{Tens}^{\mathrm{i}},$$

where the second map is the projection onto trees where all the binary vertices are above the unary ones, and the first map is the inverse of the analogous map for the binary vertices below the unary ones. The image of $(\mu^{\mathrm{i}}_{U,V}; (\iota^{\mathrm{i}})^{U}_{U'}, \mathrm{id})$ under the first map is

$$(\mu^{\mathrm{i}}_{U,V}; (\iota^{\mathrm{i}})^{U}_{U'}, \mathrm{id}) - ((\iota^{\mathrm{i}})^{U\sqcup V}_{U'\sqcup V}; \mu^{\mathrm{i}}_{U',V}).$$

Hence,

$$\Lambda^c(\mu^{\mathrm{i}}_{U,V}; (\iota^{\mathrm{i}})^{U}_{U'}, \mathrm{id}) = -((\iota^{\mathrm{i}})^{U\sqcup V}_{U'\sqcup V}; \mu^{\mathrm{i}}_{U',V}),$$

which establishes the base case, and accounts for the sign $(-1)^{(k-1)\sum(s_i-1)}$ appearing in Equation (3.1)

The rest of the proof proceeds exactly as in [Bel18]. We make two elucidations thereof. The first is that the Koszul sign rules [LV12, Section 5.1.8] appearing in the associator for the composition product $\circ$ play an important role in the proof. In Bellier-Millès's version, these appear as the $\epsilon_{k'_j,k''_j}$ signs. Those signs appear in our proof as well; in our case, there are further signs as a result of the fact that both cogenerators have odd degree. Second, Bellier-Millès omits the induction on the number of binary cogenerators. Relevant to that proof is the fact that given any shuffle $\sigma \in \mathrm{Sh}(j_1, \ldots, j_k)$ and any partition $k = \ell_1 + \cdots + \ell_n$, σ can be written uniquely in the form:

$$\tau \cdot \sigma_1 \cdot \cdots \cdot \sigma_k,$$

where $\sigma_i \in \mathrm{Sh}(j_{\ell_{i-1}+1}, \ldots, j_{\ell_i})$ and $\tau \in \mathrm{Sh}(\sum_{i=1}^{\ell_1} j_i, \ldots, \sum_{i=\ell_{n-1}+1}^{\ell_n} j_i)$. □

Now that we have made explicit the codistributive law which defines the cooperad structure on $\mathrm{Disj}^{\mathrm{i}}$, we can also make the cooperad structure itself more explicit. More precisely, since we are interested in $\mathrm{hoDisj} = \Omega\,\mathrm{Disj}^{\mathrm{i}}$, and the cobar construction makes use only of the *infinitesimal* decomposition map of the cooperad $\mathrm{Disj}^{\mathrm{i}}$, we need only address ourselves to this issue. That is the object of the following lemma. Its statement is long because of the combinatorics of the trees involved. However, we will provide an example that will illustrate this lemma graphically.

Lemma 3.7. Let $\mathcal{U} = (\mathcal{U}_1, \dots, \mathcal{U}_k)$ be as above. Under the isomorphism of $\mathbb{S}$-modules

$$\mathrm{Tens}^{\mathsf{i}} \circ \mathrm{Opens}^{\mathsf{i}} \cong \mathrm{Disj}^{\mathsf{i}},$$

the infinitesimal decomposition map $\Delta^{(1)}_{\Lambda^c}$ is given by the equation

$$\Delta^{(1)}_{\Lambda^c}(\mu^{\mathsf{i}}_{\mathcal{U}}) = \sum_{(p,q,j,s'_i,s''_i,\sigma)} \mathrm{sgn}(\sigma)(-1)^{\epsilon(s'_i,s''_i,j,p,q)} \mu^{\mathsf{i}}_{\mathcal{U}'} \circ_j \mu^{\mathsf{i}}_{\mathcal{U}''}, \tag{3.3}$$

where the sum ranges over:

- positive indices (p, q) such that $p + q = k + 1$;
- indices j from 1 to p;
- indices s'_i, s''_i satisfying $s_i - 1 = s'_i + s''_i - 2$, $s''_i = 1$ if either $i \leq j-1$ or $i > j+q-1$;
- shuffles $\sigma \in \mathrm{Sh}(s'_1 - 1, \dots, s'_q - 1)$;

and where we let:

- $\mathcal{U}'' = (\mathcal{U}''_1, \dots, \mathcal{U}''_q)$ with $\mathcal{U}''_l = U_{(j+l-1)s_1} \subsetneq \cdots \subsetneq U_{(j+l-1)s''_l}$;
- $\mathcal{U}'_1 = U_{11} \subsetneq \cdots \subsetneq U_{1s_1}, \dots, \mathcal{U}'_{j-1} = U_{(j-1)1} \subsetneq \cdots \subsetneq U_{(j-1)s_{j-1}}$;
- $\mathcal{U}'_j = \sigma \cdot (U_{j(s''_j+1)} \subsetneq \cdots \subsetneq U_{js_j}, \dots, U_{(j+q-1)(s''_{j+q-1}+1)} \subsetneq \cdots \subsetneq U_{(j+q-1)s_{j+q-1}})$;
- $\mathcal{U}'_{j+1} = U_{(q+j)1} \subsetneq \cdots \subsetneq U_{(q+j)s_{q+j}}, \quad \dots, \quad \mathcal{U}'_p = U_{k1} \subsetneq \cdots \subsetneq U_{ks_k}$;
- $\mathcal{U}' = (\mathcal{U}'_1, \dots, \mathcal{U}'_p)$;
- $\epsilon(s'_i, s''_i, j, p, q) = (q+1)(p-j) + (q-1)\Big(\sum_{i=1}^k (s'_i - 1)\Big) + \sum_{i=1}^k \Big((s''_i - 1) \sum_{\ell=i+1}^k (s'_\ell - 1)\Big)$.

The full decomposition map is given by the equation

$$\Delta_{\Lambda^c}(\mu^{\mathsf{i}}_{\mathcal{U}}) = \sum_{\substack{(p,q_1,\dots,q_p,\\ s'_i,s''_i,\sigma_1,\dots,\sigma_p)}} \Big(\prod_{i=1}^p \mathrm{sgn}(\sigma_i)\Big)(-1)^{\delta(s',s'',q,p)}(\mu^{\mathsf{i}}_{\mathcal{U}'}; \mu^{\mathsf{i}}_{1\mathcal{U}''}, \dots, \mu^{\mathsf{i}}_{p\mathcal{U}''}), \tag{3.4}$$

where the sum is over:

- positive indices $(p, q_1, \dots, q_p)$ such that $q_1 + \cdots + q_p = k$;
- indices s'_i, s''_i satisfying $s_i - 1 = s'_i + s''_i - 2$ for $1 \leq i \leq k$;
- shuffles $\sigma_j \in \mathrm{Sh}(s'_{q_1+\cdots+q_{j-1}+1} - 1, \dots, s'_{q_1+\cdots+q_j} - 1)$;

and where we let

- ${}_j\mathcal{U}''$ $(1 \leq j \leq p)$ and $\mathcal{U}'$ be defined analogously to the infinitesimal case, and

- $\delta(s', s'', q, p) = \sum_{j=1}^{p}(q_j+1)(p-j) + \Big(\sum_{j=1}^{p}(q_j-1)\Big)\Big(\sum_{i=1}^{k}(s'_i-1)\Big) + \sum_{i<\ell}\Big((s''_i-1)(s'_\ell-1)\Big)$.

Proof. The proof is very similar to the proof of [Bel18, Proposition 2.3]. The term $(q+1)(p-j)$ is the sign that appears in the decomposition product for As^{i} [LV12, Lemma 9.1.2]. The term $(q-1)\sum(s'-1)$ appears because of the corresponding term in the codistributive law. The last term $\sum((s''-1)\sum(s'-1))$ appears for the same reasons as it does in [Bel18], i.e. from the sign rules arising in the associator for the composition product $\circ$ of $\mathbb{S}$-modules. □

4 Description of homotopy prefactorization algebras

Proposition 4.1. A hoDisj algebra is a collection of spaces $\mathcal{A}(U)$, one for every open subset $U \subseteq M$, equipped with maps:

$$\mu_{\mathcal{U}} : \mathcal{A}(U_{11}) \otimes \cdots \otimes \mathcal{A}(U_{k1}) \to \mathcal{A}(U_{1s_1} \sqcup \cdots \sqcup U_{ks_k})$$

for every collection $\mathcal{U} = (U_{11} \subset \cdots \subset U_{1s_1}, \ldots, U_{k1} \subset \cdots \subset U_{ks_k})$ as in Lemma 3.3, such that:

1. The $\mu_{\mathcal{U}}$ have degree $2 - k - \sum_i(s_i - 1)$.
2. The $\mu_{\mathcal{U}}$ vanish on graded sums of shuffle permutations, i.e.

$$\sum_{\sigma \in \mathrm{Sh}(\ell_1,\ldots,\ell_n) \subset \mathbb{S}_k} \delta_\sigma\, \mu_{\mathcal{U}\cdot\sigma} \circ \sigma = 0,$$

   where:
   - σ is understood as a map

$$\mathcal{A}(U_{11}) \otimes \cdots \otimes \mathcal{A}(U_{k1}) \to \mathcal{A}(U_{\sigma^{-1}(1)1}) \otimes \cdots \otimes \mathcal{A}(U_{\sigma^{-1}(k)1}),$$

   - The symbol $\mathcal{U}\cdot\sigma$ has the meaning

$$\mathcal{U}\cdot\sigma = (\mathcal{U}_{\sigma^{-1}(1)}, \ldots, \mathcal{U}_{\sigma^{-1}(k)}),$$

   - The sign δ_σ is determined by the sign incurred from the obvious action of σ on

$$\Lambda^{\mathrm{top}}\mathbb{R}^k \otimes \Lambda^{\mathrm{top}}\mathbb{R}^{s_1-1} \otimes \cdots \otimes \Lambda^{\mathrm{top}}\mathbb{R}^{s_k-1}$$

3. The $\mu_{\mathcal{U}}$ satisfy the following relations:

$$d(\mu_{\mathcal{U}}) = \sum_{i=1}^{k}(-1)^{k+\sum_{j=1}^{i-1}(s_j-1)}\mu_{d_i\mathcal{U}} + \sum_{\substack{(p,q_1,\ldots,q_p,\\ s'_i,s''_i,\sigma_1,\ldots,\sigma_p)}} \mathrm{sgn}(\sigma)(-1)^{p-1+\sum(s'_i-1)}(-1)^{\epsilon(s'_i,s''_i,j,p,q)}\mu_{\mathcal{U}'} \circ_j \mu_{\mathcal{U}''}, \quad (4.1)$$

   where the second sum ranges over:

- positive indices $(p, q_1, \ldots, q_p)$ such that $q_1 + \cdots + q_p = k$;
- indices s_i', s_i'' satisfying $s_i - 1 = s_i' + s_i'' - 2$ for $1 \le i \le k$;
- shuffles $\sigma_j \in \mathrm{Sh}(s'_{q_1+\cdots+q_{j-1}+1} - 1, \ldots, s'_{q_1+\cdots+q_j} - 1)$.

Proof. A hoDisj-algebra is, by definition, an algebra over the colored operad $\Omega\,\mathrm{Disj}^{\mathrm{i}}$, which is semi-free on the shift of the reduced cooperad $\overline{\mathrm{Disj}}^{\mathrm{i}}$. Hence, a hoDisj-algebra has one operation for every non-identity cooperation in $\mathrm{Disj}^{\mathrm{i}}$. The operad $\Omega\,\mathrm{Disj}^{\mathrm{i}}$ has a differential which is the sum of two terms: one induced from the differential on $\mathrm{Disj}^{\mathrm{i}}$ (Equation (3.2)), and one induced from the cooperadic structure (Lemma 3.7). These terms correspond, respectively, to the two separate sums in Condition 3 of the Proposition.

The only thing which has not been spelled out in the preceding propositions is Condition 2. To establish this condition, we note that because Tens resembles a commutative operad, the sum over shuffles should be unsurprising. The main differences from the usual commutative operad are the presence of colors and of the extra composition factor Opens in Disj. We deal with the first issue by permuting also the colors (i.e. by introducing the $\mathcal{U} \cdot \sigma$). We deal with the second issue by introducing the extra s_i-dependent signs in δ_σ. ☐

Now that we have made explicit what it is to have an algebra over the operad hoDisj, we can also say what we mean by an infinity-morphism of hoDisj-algebras.

Definition 4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over the operad hoDisj. By the Rosetta Stone of the theory of homotopy algebras [LV12, Theorem 10.1.13], these are given by codifferentials $D_{\mathcal{A}}$, $D_{\mathcal{B}}$ on the cofree $\mathrm{Disj}^{\mathrm{i}}$-algebras $\mathrm{Disj}^{\mathrm{i}}(\mathcal{A}), \mathrm{Disj}^{\mathrm{i}}(\mathcal{B})$, respectively. An **infinity-morphism** of hoDisj-algebras $\mathcal{A} \rightsquigarrow \mathcal{B}$ is a map of dg-$\mathrm{Disj}^{\mathrm{i}}$-coalgebras

$$(\mathrm{Disj}^{\mathrm{i}}(\mathcal{A}), D_{\mathcal{A}}) \to (\mathrm{Disj}^{\mathrm{i}}(\mathcal{B}), D_{\mathcal{B}})$$

of the underlying semi-cofree $\mathrm{Disj}^{\mathrm{i}}$-coalgebras.

The above definition is compact, but not so useful in making explicit what one needs to check in practice to guarantee that one has an infinity-morphism. The below proposition aims to remedy this situation:

Proposition 4.3. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over the operad hoDisj, with operations $\mu_{\mathcal{U}}^{\mathcal{A}}$ and $\mu_{\mathcal{U}}^{\mathcal{B}}$, respectively, An infinity-morphism $\mathcal{A} \rightsquigarrow \mathcal{B}$ is given by a collection of maps

$$f_{\mathcal{U}} : \mathcal{A}(U_{11}) \otimes \cdots \otimes \mathcal{A}(U_{k1}) \to \mathcal{B}(U_{1s_1} \sqcup \cdots \sqcup \mathcal{B}(U_{1s_k}))$$

of degree $1 - k - \sum_i (s_i - 1)$ satisfying the symmetry property 2 in Proposition 4.1 and the relation

$$\begin{aligned} d(f_{\mathcal{U}}) = &\sum_{i=1}^{k} (-1)^{k+1+\sum_{j=1}^{i-1} s_j - 1} f_{d_i \mathcal{U}} \\ + &\sum_{(p,q,j,s_i,s_i'',\sigma)} \operatorname{sgn}(\sigma)(-1)^{p-1+\sum(s_i'-1)+\epsilon(s_i',s_i'',j,p,q)} f_{\mathcal{U}'} \circ_j \mu^A_{\mathcal{U}''} \\ - &\sum_{(p,s_i',s_i'',q_j,\sigma_j)} \Big(\prod_{j=1}^{p} \operatorname{sgn}(\sigma_j)\Big)(-1)^{\delta(s_i',s_i'',q_j,p)} (\mu^B_{\mathcal{U}'}; f_{1}\mathcal{U}'', \dots f_{p}\mathcal{U}'') \end{aligned} \tag{4.2}$$

Proof. This proposition follows from the explicit description of the cooperad $\mathsf{Disj}^{\mathrm{i}}$ and the discussion of cylinder objects in [Fre09] (cf. Figure 10 therein). $\square$

The following statement follows from the general facts of Koszul theory (cf. [LV12], Theorem 10.3.1)

Proposition 4.4. Let $\mathcal{A}$ be a hoDisj algebra and suppose given, for every open set $U \subset M$, a deformation retraction

$$\mathcal{B}(U) \underset{p_U}{\overset{i_U}{\rightleftarrows}} \mathcal{A}(U) \circlearrowleft h_U \; ;$$

then, there exists a hoDisj structure on the collection $\{\mathcal{B}(U)\}$ such that the maps i_U extend to an ∞-morphism $\mathcal{B} \rightsquigarrow \mathcal{A}$.

5 Examples

In this section, we apply the general theory of the preceding sections to some examples. First, we discuss what it means to give a hoDisj-algebra on some simple topological spaces. Next, we discuss a result concerning a factorization algebra on $\mathbb{R}$, extending a result of Costello–Gwilliam [CG17].

5.1 Preliminary: (∞-)modules over an algebra over an operad

To make explicit the descriptions of homotopy prefactorization algebras on several finite topological spaces, we will need to recall a few background notions.

Definition 5.1 (See e.g. [LV12, Section 12.3.1])**.** Let $\mathcal{N}$ be a symmetric sequence and A, M be dg-modules. The linearized composition product is defined by:

$$\mathcal{N} \circ (A; M) := \bigoplus_{n \geq 0} \mathcal{N}(n) \otimes_{\Sigma_n} \Big(\bigoplus_{i=1}^{n} A^{\otimes i-1} \otimes M \otimes A^{\otimes n-i} \Big).$$

Suppose now that $\mathcal{P}$ is a dg-operad and A is a $\mathcal{P}$-algebra. An A-$\mathcal{P}$-module (or simply A-module if $\mathcal{P}$ is obvious from the context) is an object M equipped with a map

$$\gamma_M : \mathcal{P} \circ (A; M) \to M$$

making the obvious diagrams commute.

Example 5.2*:* Any $\mathcal{P}$-algebra A is canonically a $\mathcal{P}$-module over itself. ◊

Example 5.3*:* Let $\mathcal{P}_\infty = \Omega\,\mathrm{As}^{\mathrm{i}}$ be the A_∞-operad and A be an A_∞-algebra. Then an A-module is a dg-module M equipped with maps

$$\mu_{k,i} : A^{\otimes i-1} \otimes M \otimes A^{k-i} \to M[k-2], \qquad \text{for } k \geq 2 \text{ and } 1 \leq i \leq k.$$

These maps satisfy relations such that

$$\begin{aligned} \mu_{2,1} : M \otimes A &\to M, & \mu_{2,2} : A \otimes M &\to M, \\ m \otimes a &\mapsto m \cdot a, & a \otimes m &\mapsto a \cdot m, \end{aligned}$$

endow M with a structure of an A-module up to homotopy. For example, if we write $a \otimes b \mapsto a * b$ for the binary operation in A, then we have the following relations (see Figure 4 for a graphical representation):

$$\begin{aligned} (\partial\mu_{3,1})(m, a, b) &= m \cdot (a * b) - (m \cdot a) \cdot b, \\ (\partial\mu_{3,2})(a, m, b) &= a \cdot (m \cdot b) - (a \cdot m) \cdot b, \\ (\partial\mu_{3,3})(a, b, m) &= (a * b) \cdot m - a \cdot (b \cdot m). \end{aligned}$$ ◊

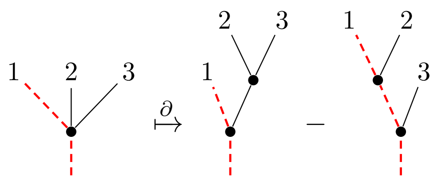


Figure 4: The first relation of Example 5.3 illustrated by trees. The solid black edges are colored by A, the dashed red colored by M.

This notion could have been defined using the moperad [Wil16, Definition 9] given by the shift $\mathcal{P}(\_ + 1) = \{\mathcal{P}(r+1)\}_{r \geq 0}$. While we couldn't specifically find the next definition in the existing literature, it follows directly from the moperadic description (see [LV12, Section 10.2.2] for the unicolored case).

Proposition/Definition 5.4. Let $\mathcal{P}_\infty = \Omega\mathcal{C}$ be the cobar construction of a dg-cooperad, let $\kappa : \mathcal{P}_\infty \to \mathcal{C}$ be the canonical Koszul twisting morphism, let A, A' be $\mathcal{P}_\infty$-algebras, and let M, M' be modules over these algebras. An ∞-morphism $(f, g) : (A, M) \rightsquigarrow (A', M')$ is the data of:

- An ∞-morphism of $\mathcal{P}_\infty$-algebras $f : A \rightsquigarrow A'$, i.e., a morphism of dg-$\mathcal{C}$-coalgebras between the bar constructions $B_\kappa A = (\mathcal{C}(A), d_\kappa) \to B_\kappa A' = (\mathcal{C}(A'), d_\kappa)$. Such a morphism is uniquely determined by a map $f : \mathcal{C} \circ A \to A'$ satisfying several compatibility relations.

- A map $g : \mathcal{C} \circ (A; M) \to M'$ which is compatible with f and the algebra/module structures. This map has to satisfy the following relations. Suppose that

  $$\xi = x(a_1, \dots, a_{i-1}, m, a_{i+1}, \dots, a_r) \in \mathcal{C}(A; M)$$

  is an element in the linearized composition product. Let $\Delta_{(1)}(x) = \sum_\alpha x'_\alpha \circ_{j_\alpha} x''_\alpha$ be an expression of the infinitesimal decomposition product applied to x, and let $\Delta(x) = \sum_\beta y'_\beta(y''_{\beta,1}, \dots, y''_{\beta,k_\beta})$ be one for the total decomposition product. Then one has (up to signs that we will not write down here):

  $$(\partial g)(\xi) = \sum_\alpha \begin{cases} \pm(g(x'_\alpha) \circ_{j_\alpha} f(x''_\alpha))(a_1, \dots, m, \dots, a_r), & \text{if } j_\alpha \leq m \leq j_\alpha + |x''_\alpha| \\ \pm(g(x'_\alpha) \circ_{j_\alpha} g(x''_\alpha))(a_1, \dots, m, \dots, a_r), & \text{otherwise;} \end{cases}$$
  $$- \sum_\beta \pm(g(y'_\beta))(f(y''_{\beta,1}), \dots, g(y''_{\beta,i}), \dots, f(y''_{\beta,k_\beta}))(a_1, \dots, m, \dots, a_r).$$

Example 5.5*:* Let us use the notation of Example 5.3. Let A, A' be A_∞-algebras, and M, M' be an A-module (resp. A'-module). Then an ∞-morphism $(f, g) : (A, M) \rightsquigarrow (A', M')$ is the data of an ∞-morphism $f = \{f_k : A^{\otimes k} \to A'[k-1]\}_{k \geq 1}$ of A_∞-algebras and a collection of maps

$$g_{k,i} : A^{\otimes i-1} \otimes M \otimes A^{k-i} \to M'[1-k], \qquad \text{for } k \geq 1 \text{ and } 1 \leq i \leq k.$$

These maps have to satisfy various compatibility relations with the f_k and the differential, such that $g_{1,1} : M \to M'$ is a morphism of bimodules up to homotopy. For example, we have (using the notation of Example 5.3):

$$(\partial g_{2,1})(m, a) = g_{1,1}(m) \cdot f_1(a) - g_{1,1}(m \cdot a),$$
$$(\partial g_{1,2})(a, m) = f_1(a) \cdot g_{1,1}(m) - g_{1,1}(a \cdot m). \qquad \lozenge$$

5.2 Homotopy prefactorization algebras on a few finite topological spaces

Let us now describe explicitly hoDisj-algebras on a few finite topological spaces.

Remark 5.6*:* While prefactorization algebras on finite spaces appear infrequently, this section is not just an exercise in abstraction. Given a manifold X and a hoDisj_X-algebra $\mathcal{A}$, the structure maps associated to the sequence of inclusions $\emptyset \subset U \subset X$ for an open subset U satisfy relations analogous to the ones in hoDisj_S, where S is the Sierpiński space of Lemma 5.11. One can get a sense of the relationship by considering the quotient map $\pi : X \to S$ that collapses U to $\{1\}$ and $X \setminus U$ to $\{2\}$, and pushing forward structure maps along π. More complex configurations of open sets can be described in a similar way using larger finite topological spaces. $\lozenge$

Lemma 5.7. A hoDisj-algebra on $\emptyset$ is a C_∞-algebra. An infinity-morphism of hoDisj-algebras on $\emptyset$ is an infinity-morphism of C_∞-algebras.

Proof. This is almost immediate. The generators of hoDisj are all of the form:

$$m_k \coloneqq \mu^{\text{¡}}_{\underbrace{\emptyset,\dots,\emptyset}_{k \text{ times}}}.$$

These correspond to the usual generators of the operad C_∞. They vanish on signed shuffles and their differential (Equation (4.1)) is exactly the one in C_∞. □

Remark 5.8*:* By restriction, if $\mathcal{A}$ is a homotopy prefactorization algebra on any space M, then $\mathcal{A}(\emptyset)$ is naturally endowed with a C_∞-structure. It is common to normalize prefactorization algebras $\mathcal{A}$ by requiring that $\mathcal{A}(\emptyset)$ is the ground field. ◊

Let us now deal with the next simplest case, that of a singleton.

Definition 5.9. Let $\mathcal{P}_\infty = \Omega\mathcal{C}$ be the cobar construction on a cooperad, and let A be a $\mathcal{P}_\infty$-algebra. A **pointed** A-module is an A-$\mathcal{P}_\infty$-module M equipped with an ∞-module morphism $A \rightsquigarrow M$.

Lemma 5.10. Let $X = \{1\}$ be a singleton (or more generally, a nonempty indiscrete space). A hoDisj-algebra on X is a triple $\mathcal{A} = (A, M, \eta_M)$ where:

- $\mathcal{A}(\emptyset) = A$ is a C_∞-algebra;
- $\mathcal{A}(X) = M$ is a C_∞-A-module;
- $\eta_M = \mathcal{A}(\emptyset \subset X)$ is a pointing $(\mathrm{id}_A, \eta_M) : (A, A) \rightsquigarrow (A, M)$.

An infinity-morphism of hoDisj-algebras on X is an ∞-C_∞-module morphism $(A, M) \rightsquigarrow (A', M')$ that commutes with the pointings.

Proof. Let $\mathcal{A}$ be a hoDisj-algebra on X and let $A = \mathcal{A}(\emptyset)$, $M = \mathcal{A}(X)$. By Remark 5.8, the hoDisj-structure on $\mathcal{A}$ endows A with a C_∞-algebra structure, using the generators $\mu^{\text{¡}}_{\emptyset,\dots,\emptyset}$. The remaining generators are of two kinds:

1. The generators $m_{k,i} \coloneqq \mu^{\text{¡}}_{\mathcal{U}[k,i]}$, where:

$$\mathcal{U}[k,i] = \underbrace{(\emptyset,\dots,\emptyset, \overbrace{X}^{i\text{th position}}, \emptyset,\dots,\emptyset)}_{k \text{ terms}}.$$

   These correspond to maps $m_{k,i} : A \otimes \dots \otimes A \otimes M \otimes A \otimes \dots \otimes A \to M$ which endow M with a C_∞-module structure.

2. The generators $f_{k,i} \coloneqq \mu^{\text{¡}}_{\mathcal{U}'[k,i]}$, where:

$$\mathcal{U}'[k,i] = \underbrace{(\emptyset,\dots,\emptyset, \overbrace{\emptyset \subset X}^{i\text{th position}}, \emptyset,\dots,\emptyset)}_{k \text{ terms}}.$$

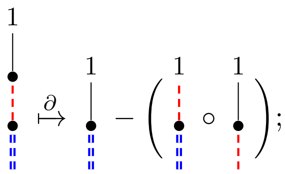
These correspond exactly the the C_∞-module structure maps and the ∞-C_∞-module morphism maps. It is a simple exercise to check that the differential from Proposition 4.1 are identical to the ones from the definition of C_∞-modules. □

The proof of the next lemma is just as straightforward, if tedious.

Lemma 5.11. Let S be the Sierpiński space, with points $\{1,2\}$ and opens $\{\emptyset, \{1\}, \{1,2\}\}$. A hoDisj-algebra on S is a septuple $\mathcal{A} = (A, M, \eta_M, N, \eta_N, f, h)$ where:

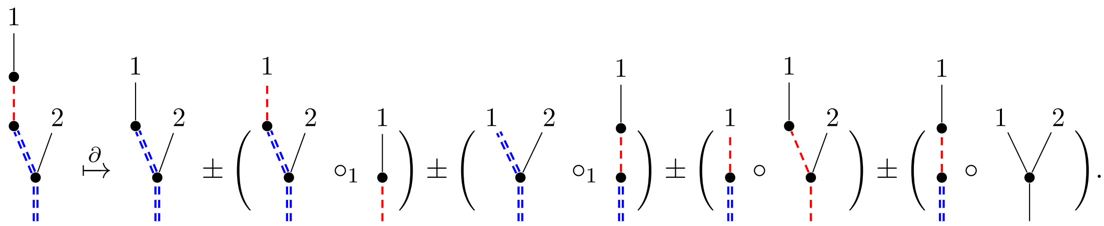
- $A = \mathcal{A}(\emptyset)$ is a C_∞-algebra;
- $M = \mathcal{A}(\{1\})$ is a C_∞-A-module;
- $\eta_M = \mathcal{A}(\emptyset \subset \{1\})$ is a pointing $(\mathrm{id}_A, \eta_M) : (A, A) \rightsquigarrow (A, M)$;
- $N = \mathcal{A}(\{1,2\})$ is a C_∞-A-module;
- $\eta_N = \mathcal{A}(\emptyset \subset \{1,2\})$ is a pointing $(\mathrm{id}_A, \eta_N) : (A, A) \rightsquigarrow (A, N)$;
- $f = \mathcal{A}(\{1\} \subset \{1,2\})$ is an ∞-C_∞-module morphism $(A, M) \rightsquigarrow (A, N)$;
- $h = \mathcal{A}(\emptyset \subset \{1\} \subset \{1,2\})$ is an ∞-module ∞-homotopy $f \circ \eta_M \simeq \eta_N : A \rightsquigarrow N$.

Example 5.12*:* Given a hoDisj_S-algebra $(A, M, \eta_M, N, \eta_N, f, h)$, the homotopy h induces the next relation from which $f \circ \eta_M$ and η_N are equal in cohomology. This relation is illustrated by the next picture, where solid black edges are colored by A, red dashed edges by M, and blue double-dashed edges by N.



or algebraically: $(\partial h)(a) = \eta_N(a) - f(\eta_M(a))$,

But there are of course other relations, such as:



Algebraically, this becomes (where we again avoid indices):

$$(\partial h)(a, b) = \eta_N(a, b) \pm f(\eta_M(a), b) \pm \mu_N(h(a), b) \pm f(\eta_N(a, b)) \pm h(\mu_A(a, b)). \qquad \diamondsuit$$

5.3 The prefactorization algebra on $\mathbb{R}$ associated to a dg Lie algebra

Let $\mathfrak{g}$ be a Lie algebra. Costello–Gwilliam [CG17] discussed a factorization algebra $\widetilde{\mathcal{F}}_{\mathfrak{g}}$ on $\mathbb{R}$ whose cohomology factorization algebra is the factorization algebra $\mathcal{F}_{\mathfrak{g}}$ associated to the universal enveloping algebra $U(\mathfrak{g})$. Explicitly,

$$\widetilde{\mathcal{F}}_{\mathfrak{g}}(U) = C_\bullet(\Omega_c(U) \otimes \mathfrak{g}),$$

where the Chevalley-Eilenberg chains $C_\bullet(\_)$ are computed using the bornological tensor product (see [CG17, Section 3.4] for details). In the aforementioned reference, it is shown that

$$H^\bullet(\widetilde{\mathcal{F}}_{\mathfrak{g}}) \cong \mathcal{F}_{\mathfrak{g}},$$

and, in particular, the cohomology is concentrated in cohomological degree 0. Since $\tilde{\mathcal{F}}_{\mathfrak{g}}$ is concentrated in non-positive cohomological degree, there is a natural quasi-isomorphism $\tilde{\mathcal{F}}_{\mathfrak{g}} \to H^\bullet(\tilde{\mathcal{F}}_{\mathfrak{g}})$. A minor modification of these arguments applies also in the case that $\mathfrak{g}$ is a graded Lie algebra. Our techniques make it straightforward to describe a map the other way, and in fact we can show that the theory applies equally well for dg Lie algebras:

Lemma 5.13. Let $(\mathfrak{g}, d_{\mathfrak{g}}, [\_,\_])$ be a dg Lie algebra. There is an infinity-quasi-isomorphism of hoDisj-algebras:

$$\mathcal{F}_{\mathfrak{g}} \rightsquigarrow \tilde{\mathcal{F}}_{\mathfrak{g}}.$$

Proof. Let $U \subseteq \mathbb{R}$ be open. Let d_{dR} denote the de Rham differential on the dg Lie algebra $\Omega^\bullet_c(U) \otimes \mathfrak{g}$. The complex $\tilde{\mathcal{F}}_{\mathfrak{g}}(U)$ has a differential induced from d_{dR}, a differential induced from $d_{\mathfrak{g}}$, and finally the Chevalley-Eilenberg differential d_{CE} (note that we use the term d_{CE} to denote only the part of the Chevalley-Eilenberg differential which is not a derivation for the symmetric algebra structure). There is a deformation retraction

$$(\textstyle\bigoplus_{\pi_0(U)} \mathfrak{g}, d_{\mathfrak{g}}) \underset{p_U}{\overset{i_U}{\rightleftarrows}} (\Omega^\bullet_c(U) \otimes \mathfrak{g}[1], d_{dR} + d_{\mathfrak{g}}) \circlearrowleft h_U\ ,$$

where p_U is, connected-component-by-connected-component, the integration map. To define the maps i_U and h_U, one makes a choice of compactly-supported one-form on $(-1/2, 1/2)$ with integral 1. This is a fairly standard deformation-retraction; see, e.g. [BT82, Section 4.3]. This deformation retraction extends to a retraction

$$(\textstyle\bigotimes_{\pi_0(U)} \operatorname{Sym} \mathfrak{g}, d_{\mathfrak{g}}) \underset{p_U}{\overset{i_U}{\rightleftarrows}} (\operatorname{Sym}(\Omega^\bullet_c(U) \otimes \mathfrak{g}[1]), d_{dR} + d_{\mathfrak{g}}) \circlearrowleft h_U$$

satisfying the side conditions $h_U^2 = p_U h_U = h_U i_U = 0$. The complex $\tilde{\mathcal{F}}_{\mathfrak{g}}$ is a perturbation of the right-hand side of the above retraction, namely by the differential d_{CE}. The homological perturbation lemma [Cra04] applies in this situation: the differential d_{CE} lowers sym-degree by 1 and h_U preserves sym-degree, so the infinite sum $\sum_{n=0}^\infty (d_{CE} h_U)^n$ is well-defined on any symmetric tensor of fixed sym-degree. Hence, we obtain a deformation retraction

$$(\textstyle\bigotimes_{\pi_0(U)} \operatorname{Sym} \mathfrak{g}, d_{\mathfrak{g}} + \delta) \underset{p'_U}{\overset{i'_U}{\rightleftarrows}} \tilde{\mathcal{F}}_{\mathfrak{g}} \circlearrowleft h'_U\ ,$$

where

$$i'_U = \sum_{n=0}^\infty (h_U d_{CE})^n i_U, \quad p'_U = \sum_{n=0}^\infty p_U (d_{CE} h_U)^n, \quad h'_U = \sum_{n=0}^\infty h_U (d_{CE} h_U)^n$$
$$\delta = p_U \sum_{n=0}^\infty (\eta_U d_{CE})^n i_U.$$

The differential d_{CE} is trivial on the image of i_U; hence, $\delta = 0$. For the same reason, we get $i'_U = i_U$. The homotopy transfer theorem 4.4 now guarantees the existence of a hoDisj-algebra structure on the collection of graded vector spaces $\bigotimes_{\pi_0(U)} \mathrm{Sym}\, \mathfrak{g}$ (as U ranges over the open subsets of $\mathbb{R}$), and an infinity-morphism from this hoDisj-algebra to $\widetilde{\mathcal{F}}_{\mathfrak{g}}$. Note that, by the PBW theorem, $\bigotimes_{\pi_0(U)} \mathrm{Sym}(\mathfrak{g}) \cong \bigotimes_{\pi_0(U)} U(\mathfrak{g}) = \mathcal{F}_{\mathfrak{g}}(U)$.

Our goal now is to show that this hoDisj-algebra structure coincides with the Disj-algebra structure on $\mathcal{F}_{\mathfrak{g}}$ constructed from the associative product on $U(\mathfrak{g})$. Since the formulas for the transfered hoDisj-algebra structure on $\widetilde{\mathcal{F}}_{\mathfrak{g}}$ do not depend in any way on $d_{\mathfrak{g}}$, it suffices to consider the case that $d_{\mathfrak{g}} = 0$. Let us proceed to show that there are no operations of negative degree in the transferred hoDisj-algebra structure on $\mathcal{F}_{\mathfrak{g}}(U)$.

To this end, it is useful to introduce a grading on $\widetilde{\mathcal{F}}_{\mathfrak{g}}$ which we call the syzygy-degree, by analogy with the case of the bar and cobar constructions of a quadratic algebra/coalgebra. The syzygy degree in $\widetilde{\mathcal{F}}_{\mathfrak{g}}(U)$ is the total sym-degree minus the form degree. The syzygy degree for all elements of $\mathcal{F}_{\mathfrak{g}}(U)$ is zero. With these choices, the maps i_U, p_U, h_U, and d_{CE} have syzygy degrees 0, 0, +1, and −1, respectively. It follows that i'_U, p'_U, and h'_U have the same degrees as their un-primed counterparts. Moreover, the Disj-operations on $\widetilde{\mathcal{F}}_{\mathfrak{g}}$ have syzygy degree 0.

The operations of the hoDisj-algebra structure on $\mathcal{F}_{\mathfrak{g}}$ induced from homotopy transfer can be represented by a sum of trees with vertices of valence two or three. The leaves of each tree are labeled by i'_U (possibly for varying values of U), the internal edges are labeled by h'_U, and the root is labeled by p'_U. The vertices are labeled by operations from $\widetilde{\mathcal{F}}_{\mathfrak{g}}$. The total syzygy degree of such a tree is just the number of internal edges; since $\mathcal{F}_{\mathfrak{g}}$ is concentrated in syzygy degree 0, it follows that no trees with internal edges contribute to the hoDisj-algebra structure on $\mathcal{F}_{\mathfrak{g}}$. But the trees with no internal edges are precisely the ones that induce the Disj-algebra structure which $\mathcal{F}_{\mathfrak{g}}$ obtains as the cohomology of $\widetilde{\mathcal{F}}_{\mathfrak{g}}$. This Disj-algebra structure is precisely the one associated to the universal enveloping algebra $U(\mathfrak{g})$ [CG17]. □

References

[Bel18] Joan Bellier-Millès. "Complex manifolds as families of homotopy algebras". In: *Rend. Semin. Mat. Univ. Padova* 140 (2018), pp. 129–172. ISSN: 0041-8994. DOI: 10.4171/RSMUP/140-6.

[BT82] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*. Graduate Texts in Mathematics 82. New York-Berlin: Springer-Verlag, 1982. 331 pp. ISBN: 0-387-90613-4. DOI: 10.1007/978-1-4757-3951-0.

[CW15] Damien Calaque and Thomas Willwacher. "Triviality of the higher Formality Theorem". In: *Proc. Amer. Math. Soc.* 143.12 (2015). ISSN: 0002-9939. DOI: 10.1090/proc/12670. eprint: 1310.4605 (arXiv).

[CFM21] Victor Carmona, Ramon Flores, and Fernando Muro. *A model structure for locally constant factorization algebras*. 2021. arXiv: 2107.14174. Prepublished.

[CG17] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory*. Vol. 1. New Mathematical Monographs 31. Cambridge: Cambridge University Press, 2017. 387 pp. ISBN: 978-1-107-16310-2. DOI: 10.1017/9781316678626.

[CG21] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory*. Vol. 2. New Mathematical Monographs 41. Cambridge: Cambridge University Press, 2021. 402 pp. ISBN: 978-1-107-16315-7; 978-1-009-00616-3. DOI: 10.1017/9781316678664.

[Cra04] M. Crainic. *On the perturbation lemma, and deformations*. 2004. arXiv: math/0403266. Pre-published.

[Fre09] Benoit Fresse. "Operadic cobar constructions, cylinder objects and homotopy morphisms of algebras over operads". In: *Alpine perspectives on algebraic topology*. Contemp. Math. 504. Providence, RI: AMS, 2009, pp. 125–188. DOI: 10.1090/conm/504/09879.

[HM12] Joseph Hirsh and Joan Millès. "Curved Koszul duality theory". In: *Mathematische Annalen* 354.4 (Dec. 1, 2012), pp. 1465–1520. DOI: 10.1007/s00208-011-0766-9. URL: https://doi.org/10.1007/s00208-011-0766-9.

[HKR62] Gerhard Hochschild, Bertram Kostant, and Alex Rosenberg. "Differential forms on regular affine algebras". In: *Trans. Amer. Math. Soc.* 102 (1962), pp. 383–408. ISSN: 0002-9947.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*. Grundlehren der mathematischen Wissenschaften 346. Berlin-Heidelberg: Springer, 2012. 634 pp. ISBN: 978-3-642-30361-6. DOI: 10.1007/978-3-642-30362-3.

[Wil16] Thomas Willwacher. "The Homotopy Braces Formality Morphism". In: *Duke Math. J.* 165.10 (2016), pp. 1815–1964. DOI: 10.1215/00127094-3450644. arXiv: 1109.3520.