Conclusion (Erik Lindell)

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S_g^b : compact oriented surface of genus g w/ b boundary components

$\mathrm{Mod}(S_g^b) = \pi_0 \mathrm{Diff}^+_\partial(S_g^b)$: mapping class group

$\mathcal{I}_g = \ker\left(\mathrm{Mod}(S_g) \longrightarrow GL(H_1(S_g;\mathbb{Z}))\right)$: Torelli group

short exact sequence : $1 \longrightarrow \mathcal{I}_g \longrightarrow \mathrm{Mod}(S_g) \longrightarrow Sp_{2g}(\mathbb{Z}) \longrightarrow 1$

$\Rightarrow H_*(\mathcal{I}_g;\mathbb{Q})$ is a representation of $Sp_{2g}(\mathbb{Z})$

Thm [Minahan] For $g \geq 51$, $H_2(\mathcal{I}_g;\mathbb{Q})$ is finite dim

First main tool : representations of transvective type

$\rightarrow$ given a gp G, finite set F of pairwise commuting elements, V : rep of G

for any $F' \subsetneq F$, choose $C'_G(F') \leq C_G(F)$, such that : (set $C'_G(\emptyset) = G$)

- $\forall F' \subseteq F''$, $C'_G(F') \geq C'_G(F'')$
- $\forall F' \subsetneq F$, $\forall f \in F \setminus F'$, we have $\underbrace{\mathrm{ncl}}_{\text{normal closure}}{}_{C'_G(F')}(f) = C'_G(F')$

(i) • For any $F' \subsetneq F$, $H_0(C'_G(F');V)$ is fdim

(ii) • $\mathrm{coker}\left(\bigoplus_{f\in F} V^f \longrightarrow V\right)$ is fdim

(iii) • For any $F' \subsetneq F$, $C'_G(F')$ is gen by a finite subset of $\bigcup_{h \in C'_G(F')} h(F \setminus F')h^{-1}$

Thm If (G, F, V) is of transvective type, then V is finite dimensional

Goal Show that $(Sp_{2g}(\mathbb{Z}), \{T_{[a]}\}, H_2(\mathcal{I}_g;\mathbb{Q}))$ is of transvective type

where $T_{[a]} \in Sp_{2g}(\mathbb{Z})$ is a primitive transvection, i.e. the image of a Dehn twist T_a along a simple closed non-separating curve in S_g

Prop Let $g \geq 2$, $b \geq 0$, $M \subseteq S_g^b$ a non-separating multicurve st $g(S_g^b \setminus M) \geq 1$. Let G be the image of the composition

$$\mathrm{Mod}(S_g^b \setminus M) \longrightarrow \mathrm{Mod}(S_g^b) \longrightarrow \mathrm{Aut}(H_1(S_g^b;\mathbb{Z}))$$

Then (a) G has a finite generating set of transvections

(b) For any primitive transvection $T_{[a]} \in G$, $\mathrm{ncl}(T_{[a]}) = G$

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(b) For any primitive transvection $T_{[a]} \in G$, $\mathrm{ncl}(T_{[a]}) = G$

(a) implies that $(Sp_{2g}(\mathbb{Z}), \{T_{[a]}\}, H_2(\mathcal{I}_g))$ satisfies (iv)
(b) ———————————————————————— (ii)

Prop For $g \geq 3$, $H_0(Sp_{2g}(\mathbb{Z}); H_2(\mathcal{I}_g; \mathbb{Q}))$ is fdim

Pf Let E be the Hochschild-Serre spectral sequence associated to
$$1 \to \mathcal{I}_g \to \mathrm{Mod}(S_g) \to Sp_{2g}(\mathbb{Z}) \to 1$$

$$E^2_{p,q} = H_p(Sp_{2g}(\mathbb{Z}); H_q(\mathcal{I}_g; \mathbb{Q})) \Longrightarrow H_{p+q}(\mathrm{Mod}(S_{2g}); \mathbb{Q})$$

We have a surjection (edge map) $E^2_{0,2} \xrightarrow{f} E^\infty_{0,2} \subseteq \underbrace{H_2(\mathrm{Mod}(S_g); \mathbb{Q})}_{\text{fdim}}$
$\Rightarrow \dim \operatorname{im} f < \infty$

$H_q(\mathcal{I}_g; \mathbb{Q})$ is fdim for $q = 0, 1$ $\Rightarrow E^2_{p,q} = H_p(Sp_{2g}(\mathbb{Z}); H_q(\mathcal{I}_g; \mathbb{Q}))$ is fdim for $q = 0, 1$

$\Rightarrow \dim \ker f < \infty \Rightarrow E^2_{0,2} = H_0(Sp(2g, \mathbb{Z}); H_2(\mathcal{I}_g; \mathbb{Q}))$

$\Rightarrow (Sp(2g, \mathbb{Z}), \{T_{[a]}\}, H_2(\mathcal{I}_g; \mathbb{Q}))$ satisfies (ii)

It remains to verify that $\mathrm{coker}(H_2(\mathcal{I}_g; \mathbb{Q})^{T_{[a]}} \to H_2(\mathcal{I}_g; \mathbb{Q}))$ is fdim

Conjugation by T_a acts trivially on $(\mathcal{I}_g)_a = \mathrm{Stab}_{\mathcal{I}_g}(a)$

$\Rightarrow H_2((\mathcal{I}_g)_a; \mathbb{Q}) \to H_2(\mathcal{I}_g; \mathbb{Q})$ factors through $H_2((\mathcal{I}_g)^{T_a}; \mathbb{Q}) \to H_2(\mathcal{I}_g; \mathbb{Q})$

Thm $\mathrm{coker}(H_2((\mathcal{I}_g)_a; \mathbb{Q}) \to H_2(\mathcal{I}_g; \mathbb{Q}))$ is fdim

Second main tool: equivariant homology spectral sequence

Let a be a simple closed non-separating curve in S_g and $[a] \in H_1(S_g; \mathbb{Z})$
We define $C_{[a]}(S_g)$ to be the simplicial cx w/
- vertices: closed curves st $[c] = [a]$
- $(c_1, \ldots, c_k)$ forms a k-simplex $\Leftrightarrow |c_i \cap c_j| = 0$

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- $(C_1, \dots, C_k)$ forms a k-simplex $\iff |C_i \cap C_j| = 0$

Thm (Minahan 2023) For $g \geq 2$, $k \leq g-3$, then $\tilde{H}_k(C_{[a]}(S_g); \mathbb{Z}) = 0$

Since $\mathcal{I}_g \subset C_{[a]}(S_g)$ we get an equivariant homology spectral sequence

$$E^1_{p,q} = \bigoplus_{\sigma \in \mathcal{E}_p} H_p((\mathcal{I}_g)_\sigma ; \mathbb{Q}_\sigma) \Longrightarrow H^{\mathcal{I}_g}_{p+q}(C_{[a]}(S_g); \mathbb{Q})$$

where $\mathcal{E}_p$ = set of representatives of $C_{[a]}(S_g)^{(p)} / \mathcal{I}_g$ and $\mathbb{Q}_\sigma$ is the orientation representation of $(\mathcal{I}_g)_\sigma$

Some observations:

- for x, y isotopy classes of homologous curves in S_g, $\exists f \in \mathcal{I}_g$ st $fx = y$
  $\Rightarrow |\mathcal{E}_0| = 1$ and $E^1_{0,2} = H_2((\mathcal{I}_g)_a ; \mathbb{Q})$

- since $C_{[a]}(S_g)$ is 2-acyclic for $g \geq 5$, we have
  $$H_2^{\mathcal{I}_g}(C_{[a]}(S_g); \mathbb{Q}) \simeq H_2(\mathcal{I}_g ; \mathbb{Q})$$

- the composition $H_2((\mathcal{I}_g)_a ; \mathbb{Q}) = E^1_{0,2} \twoheadrightarrow E^\infty_{0,2} \hookrightarrow H_2(\mathcal{I}_g; \mathbb{Q})$ is the pushforward map

- since $H_2(\mathcal{I}_g ; \mathbb{Q}) \simeq E^\infty_{0,2} \oplus E^\infty_{1,1} \oplus E^\infty_{2,0}$,
  $\mathrm{coker}(H_2((\mathcal{I}_g)_a ; \mathbb{Q}) \to H_2(\mathcal{I}_g ; \mathbb{Q})) \simeq E^\infty_{1,1} \oplus E^\infty_{2,0}$

Thm B follows if we show that $E^2_{1,1}$ and $E^2_{2,0}$ are fdim
$\rightarrow$ proved using the Johnson homomorphism