Let $S = S_{g,1}$: surface of genus g, one ∂ component
Fix a basepoint on ∂S, $\pi = \pi_1 S$, $H = H_1 S$, $H_\mathbb{Q} = H_1(S;\mathbb{Q})$
T : Torelli group $= \ker(\mathrm{Mod}(S) \longrightarrow Sp_{2g})$

Let $X = \mathbb{C}^g / \mathbb{Z}^{2g}$: complex torus of $\dim_\mathbb{C} = g$
$X = B(\mathbb{Z}^{2g}) = K(\mathbb{Z}^{2g}, 1)$
The abelianization $\pi \longrightarrow H$ corresponds to a homotopy class of maps $S \to X$

1) The Abel-Jacobi map

Let $\overline{S}$ be the closed surface obtained by gluing back a disc to S
Choose a base point + tangent vector in the disc
Complex structure on S = pair (C, h) where C is a compact Riemann surface and h is a diffeo $C \xrightarrow{\sim} \overline{S}$ which preserves the basepoint & tgt vector

Two complex structures (C,h), (C',h') are isotopic if $h^{-1} \circ h'$ is isotopic (rel basepoint & tgt vector) to a biholomorphism

Recall that the cotangent bundle of C has a canonical structure of a holomorphic bundle $\rightsquigarrow$ canonical line bundle K
The space of global sections of K, ie $H^0(C,K)$, is the space of holomorphic 1-forms on C. Note that $H^0(C;K) \cong H^1_{dR}(C;\mathbb{R})$ as $\mathbb{R}$-vector spaces

If $\alpha \in H^0(C;K)$ and γ is a path on C, then let $\int_\gamma \alpha = \int_{[0,1]} \gamma^* \alpha$
$\Rightarrow$ induces a non-degen pairing $H^0(C,K) \otimes H_1(S;\mathbb{Z}) \longrightarrow \mathbb{C}$

$\Rightarrow H_1(S;\mathbb{Z}) \hookrightarrow H^0(C,K)^*$
The Jacobian of C is $J(C) = H^0(C,K)^* / H_1(S;\mathbb{Z})$
choosing a (symplectic) basis of H, we get $J(C) \cong \mathbb{C}^g / \mathbb{Z}^{2g} \cong X$

The Abel-Jacobi map is defined as follows:
given $y \in \overline{S}$, and γ a path from the basepoint to y, then let
$j(y) \in H^0(C,K)^*$, $\alpha \longmapsto \int_\gamma \alpha$

If γ' is another path to y, then $\gamma' \cdot \gamma^{-1}$ is a loop $\Rightarrow \int_{\gamma'\gamma^{-1}} (-) = 0$ in $J(C)$
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$\Rightarrow j(y)$ does not depend on γ
$\Rightarrow$ get a well def map $j : C \longrightarrow J(C) \simeq X$
j induces the abelianization $\pi_1 C \longrightarrow \pi_1(J(C)) = \mathbb{Z}^{2g} = H$

2) Bundles over the Torelli space

Let Teich be the space whose points are isotopy classes of complex structures on S

Thm Teich is homeomorphic to $\mathbb{R}^{6g-3}$ $(\simeq *)$
The Torelli group T acts freely and properly on Teich

The Torelli space of S is $\text{Teich}/T = \mathcal{T}$
Rk $\mathcal{T} \simeq K(T,1)$

In particular, $H_*(\mathcal{T}) \simeq H_*(T)$
Let $\mathcal{T}^* = (S \times \text{Teich})/T$: homotopy quotient $S /\!/ T$
$p : \mathcal{T}^* \longrightarrow \mathcal{T}$ be the projection
Similarly $J^* = (X \times \text{Teich})/T \simeq X \times \mathcal{T}$ ($T \curvearrowright H$ is trivial)

$\rightarrow$ the Abel-Jacobi map assembles to a map $S \times \text{Teich} \longrightarrow X \times \text{Teich}$
$\Rightarrow \mathcal{T}^* \longrightarrow J^*$ $\qquad (y,(c,h)) \longmapsto (j_{c,h}(y), c)$

Rk There is a residual action of $Sp(H) = \text{Mod}(S)/T$ on $\mathcal{T}$, and
$\mathcal{T}^* \longrightarrow J^*$ is $Sp(H)$-equivariant

We thus get a diagram : $\boxed{\mathcal{T} \longleftarrow \mathcal{T}^* \rightarrow J^* \longrightarrow X}$

Let $f : M \longrightarrow N$ be a map b/w compact oriented manifold, preserving the boundary
such that $\dim N = m$, $\dim M = n+k$. Then the pullback in homology is
the composition : $f^* : H_i(N) \underset{\text{Poincaré duality}}{\longrightarrow} H^{m-i}(N,\partial N) \longrightarrow H^{m-i}(M,\partial M) \underset{\text{Poincaré duality}}{\longrightarrow} H_{i+k}(M)$

Rk Under some transversality conditions, if $Y \subset N$, then $f^*[N] \in H_{i+k}(M)$
is the class of the preimage of Y (sub-mfd)

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The i-th geometric Johnson homomorphism is the map
$\tau_i' : H_i(\mathcal{T}) \longrightarrow H_{i+2}(X;\mathbb{Q}) = \Lambda^{i+2} H_{\mathbb{Q}}$ given by pulling back along p and pushing forward to $H_{i+2}(X)$
$= H_i(\mathcal{T})$

Thm [Johnson, Hain, Church-Farb] $\tau_1 = \tau_1'$ (τ_1 = last week's map)

Let $\sigma \in H_1(\mathcal{T})$. Assume that there is a map $B \xrightarrow{g} \mathcal{T}$ and some $x \in H_i(B)$ such that $g_*(x) = \sigma$

Let E be the pullback $E = \{(x,t) \in \mathcal{T}^* \times B \mid p(x) = g(t)\}$
To compute $\tau_i(\sigma)$, we can run the construction with $B \leftarrow E \xrightarrow{j_E} X$
In particular, if $x = [B]$, then $\tau_i(\sigma) = (j_E)_*([E])$

Let $f \in \mathcal{T}$. If we pick a basepoint in $\mathcal{T}$, then this corresponds to a loop $\gamma_f : \underbrace{S^1}_{=B} \to \mathcal{T}$
In that case, the pullback bundle is the mapping torus M_f
$[f] \in H_1(\mathcal{T})$ is the image of the fundamental class in $H_1(S^1)$
$\tau_1(f) = j_*([M_f])$

Let (α,β) be a bounding pair, $f = T_\alpha^{-1} T_\beta$
Let S' be the component of $S \setminus (\alpha \cup \beta)$ which doesn't contain the basepoint and let $\{a_i, b_i\}$ be a symplectic basis st $[\alpha] = a_{k+1}$
and $\{a_1, b_1, \dots, a_k, b_k, a_{k+1}\}$ is a basis of $H(S')$

Then $\boxed{\tau_1(f) = \Big(\underbrace{\sum_{i=1}^{k} a_i \wedge b_i}_{=\omega_{S'}}\Big) \wedge a_{k+1}} \in \Lambda^3 H_{\mathbb{Q}}$