References : Weibel : "An introduction to homological algebra"
Brown : "Cohomology of groups"

① Reminder on derived functors
② Tools for computations
③ Computations & geometrical interpretation

Plan

① Reminder on derived functors

Fix R a ring, eg $R = \mathbb{Z}[G]$, $\mathbb{Q}[G]$

def Let $F : R\text{-Mod} \to Ab$ be an additive & right exact functor

Left derived functor : family $(L_i F : R\text{-Mod} \to Ab)_{i \in \mathbb{N}}$

⇒ given A, choose a proj resolution $P_\bullet \xrightarrow{\varepsilon} A$

& set $L_i F(A) := H_i(F(P_\bullet))$

⇒ given $A \to B$, choose $P \xrightarrow{\sim} A$, $Q \xrightarrow{\sim} B$ & extend

$$\begin{array}{ccc} P_\bullet & \dashrightarrow & Q_\bullet \\ \downarrow\sim & & \downarrow\sim \\ A & \longrightarrow & B \end{array}$$

⇒ induces $L_i F(A) \to L_i F(B)$

$$\begin{array}{ccc} L_i F(A) & \longrightarrow & L_i F(B) \\ \| & & \| \\ H_i(F(P_\bullet)) & \longrightarrow & H_i(F(Q_\bullet)) \end{array}$$

Rk Everything is unique up to homotopy ⇒ well def

Right derived functor of a left exact functor $F : R\text{-Mod} \to Ab$

⇒ given $A \in R\text{-Mod}$, choose injective resolution $A \xhookrightarrow{\sim} I_\bullet$

& let $R^i F(A) := H_i(F(I_\bullet))$

⇒ given $A \to B$, choose $A \xrightarrow{\sim} I_\bullet$, $B \xrightarrow{\sim} J_\bullet$, lift to $I_\bullet \to J_\bullet$.
to define $H_i(F(I_\bullet)) \longrightarrow H_i(F(J_\bullet))$
$= R^i F(A) \longrightarrow R^i F(B)$

Rk If A is proj then $L_i F(A) = 0 \quad \forall i \geq 1$

— inj — $R_i F(A) = 0 \quad \forall i \geq 1$

def $Q \in R\text{-Mod}$ is **F-acyclic** if $L_i F(Q) = 0 \ \forall i \geq 1$
(resp $R^i F(Q) = 0 \ \forall i \geq 1$)

prop If $Q_\bullet \xrightarrow{\sim} A$ is an F-acyclic resolution then $L_i F(A) = H_i(F(Q_\bullet))$
(& dual statement for right derived functors)

Rk For all $f : A \to B$ in $R\text{-Mod}$, we have a comm square:

$$\begin{array}{ccc} L_0 F(A) & \xrightarrow{\sim} & F(A) \\ f \downarrow & & \downarrow f \\ L_0 F(B) & \xrightarrow{\sim} & F(B) \end{array}$$

Thm Let F be a right exact (resp left exact) functor and
suppose $0 \to A' \to A \to A'' \to 0$ is a short exact sequence
Then there is a natural long exact sequence:

$$\dots \to L_{n+1} F(A'') \xrightarrow{\delta} L_n F(A') \to L_n F(A) \to L_n F(A'') \xrightarrow{\delta} \dots$$

(resp: $\dots \to R^{n-1} F(A'') \xrightarrow{\delta} R^n F(A') \to R^n F(A) \to R^n F(A'') \xrightarrow{\delta} \dots$)

Pf If $P'_\bullet \to A'$ and $P''_\bullet \to A''$ are proj resolutions
then can find a proj resol° $P_\bullet \xrightarrow{\sim} A$ that fits in:

$$\begin{array}{ccccc} P' & \to & P & \to & P'' \quad \text{exact} \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ A' & \to & A & \to & A'' \end{array}$$

(horseshoe lemma)

Thm $L_\bullet F$ is terminal among $\left\{ \begin{array}{l} T_\bullet : R\text{-Mod} \to Ab \\ T_0 \to F \end{array} \middle| \begin{array}{l} T_n \text{ is additive} \\ \text{short exact} \rightsquigarrow \text{long exact} \end{array} \right\}$

$R^\bullet F$ is initial among {similar description}

Def **$\mathrm{Tor}^R_*(A, B)$** $:= L_*(A \otimes_R -)(B)$ where $A \otimes_R - : R\text{-Mod} \to Ab$ is right exact
$\mathrm{Ext}^*_R(A, B)$ $:= R^* \mathrm{Hom}_R(A, -)(B)$ where $\mathrm{Hom}_R(A,-)$ is left exact

Thm $L_*(A \otimes_R -)(B) \cong L_*(- \otimes_R B)(A) = \mathrm{Tor}^R_*(A,B)$
$R^* \mathrm{Hom}_R(-, B)(A) \simeq R^* \mathrm{Hom}_R(A, -)(B) = \mathrm{Ext}^*_R(A, B)$

Thm $L_*(A \otimes_R -)(B) \cong L_*(- \otimes_R B)(A) = \mathrm{Tor}^R_*(A,B)$

$R^* \mathrm{Hom}_R(-,B)(A) \cong R^* \mathrm{Hom}_R(A,-)(B) = \mathrm{Ext}^*_R(A,B)$

② Tools for computations

Fix k a commutative ring, G a group, A a $k[G]$-module

def $H_*(G;M) := \mathrm{Tor}_*^{k[G]}(k, A)$ where k is the trivial $k[G]$-module

$H^*(G;M) := \mathrm{Ext}_*^{k[G]}(k, A)$

Interpretation: $H_0(G;A) = k^{triv} \otimes_G A = A_G = A/(g\cdot a - a)_{g\in G, a \in A}$ coinvariants

$H^0(G;A) = \mathrm{Hom}_G(k^{triv}, A) = A^G = \{a \in A \mid \forall g, g\cdot a = a\}$ invariants

$\Rightarrow H_*(G;-)$ is the derived functor of coinvariant

$H^*(G;-)$ ———————— invariants

Thm Given a SES of G-modules $0 \to A' \to A \to A'' \to 0$, we have LES:

$$\cdots \to H_{n+1}(G;A'') \xrightarrow{\delta} H_n(G;A') \to H_n(G;A) \to H_n(G;A'') \xrightarrow{\delta} \cdots$$

$$\cdots \to H^{n-1}(G;A'') \xrightarrow{\delta} H^n(G;A') \to H^n(G;A) \to H^n(G;A'') \xrightarrow{\delta} \cdots$$

Thm [Shapiro's lemma] Let $H \leq G$ be a subgroup of G and A be an H-module

then $H_*(G; \mathrm{Ind}_H^G(A)) \cong H_*(H;A)$

$H^*(G; \mathrm{Coind}_H^G(A)) \cong H^*(H;A)$

where $\mathrm{Ind}_H^G(A) = k[G] \otimes_H A$, $\mathrm{Coind}_H^G(A) = \mathrm{Hom}_H(k[G], A)$

left / right adjoints to forgetful functor $G\text{-Mod} \to H\text{-Mod}$

proof $k[G] \cong \bigoplus_{G/H} k[H]$ is a free H-module

$\Rightarrow$ if $P_\bullet \xrightarrow{\sim} k$ is a projective resolution of k as G-modules, then it is also projective as H-modules

and $H_*(G, \mathrm{Ind}_H^G(A)) = \mathrm{Tor}_*^G(k, \mathrm{Ind}_H^G(A)) = H_*(P \otimes_G (k[G] \otimes_H A))$

and $H_*(G, \mathrm{Ind}_H^G(A)) = \mathrm{Tor}_*^G(k, \mathrm{Ind}_H^G(A)) = H_*(P \underset{G}{\otimes} (k[G] \underset{H}{\otimes} A))$
$= H_*(P \underset{H}{\otimes} A) = \mathrm{Tor}_*^H(k, A)$

Similar reasoning for cohomology

Prop If $1 \to H \to G \to G/H \to 1$ is a SES of groups, then G/H acts on $H_*(H; A)$ for all $A \in H\text{-Mod}$

<u>Functoriality of H_*</u>: let $\mathcal{C}$ be the category of pairs (G, A) | G: group, A: G-module

- If $\rho : H \to G$ is a morphism of groups, then for all $A \in G\text{-Mod}$, there is a natural surjection $(\rho^\# A)_H \longrightarrow A_G$

  $\Rightarrow$ by the universal property of derived functor, we get

  $\mathrm{Cor}_H^G = \rho_* : H_*(H; \rho^\# A) \longrightarrow H_*(G; A)$
- If $(\rho, \varphi) : (H, B) \longrightarrow (G, A)$ is a morφ in $\mathcal{C}$, we have a map $\mathrm{Cor}_H^G \circ \varphi : H_*(H; B) \to H_*(G; A)$

$\Rightarrow H_*(-;-) : \mathcal{C} \longrightarrow Ab$ is a functor

- If $H \leq G$ is a subgroup, then for all $g \in G$, we have a morphism $\rho_g : H \to gHg^{-1}$. If $A \in G\text{-Mod}$,

  $$\begin{aligned} \psi_g : A &\longrightarrow \rho_g^\# A \\ a &\longmapsto ga \end{aligned}$$ is a morphism of H-modules

  The pair (ρ_g, ψ_g) is an iso in $\mathcal{C}$

  $\Rightarrow$ we get a map $g : H_*(H; A) \xrightarrow{\simeq} H_*(gHg^{-1}; A)$

  If H is normal, then $gHg^{-1} = H \Rightarrow g : H_*(H; A) \to H_*(H; A)$ is an iso. Moreover if $g \in H$ then this is the identity

  $\Rightarrow$ get an action $G/H \circlearrowright H_*(H; A)$

<u>Rk</u> $g : H_*(H; A) \to H_*(gHg^{-1}; A)$ is induced as follows:

Choose $P \to k^{\mathrm{triv}}$ a G-proj resol°. Then $g : P \otimes A \longrightarrow P \otimes A$

Rk $g : H_*(H;A) \to H_*(gHg^{-1};A)$ is induced as follows:
Choose $P_\bullet \to k^{triv}$ a G-proj resol°, then $g : P_\bullet \otimes_H A \to P_\bullet \otimes_{gHg^{-1}} A$

$x \otimes a \mapsto gx \otimes ga$

Bar resolution

take $k = \mathbb{Z}$

the adjunction $\mathbb{Z}\text{-Mod} \underset{U}{\overset{\mathbb{Z}[G]\otimes -}{\rightleftarrows}} \mathbb{Z}[G]\text{-Mod}$

produces free bimodule. We have a free resolution of $\mathbb{Z}^{triv}$:

$$\left(\cdots \to B_2^u G \to B_1^u G \to B_0^u G\right) \xrightarrow[\varepsilon]{\sim} \mathbb{Z}$$

$B_0^u G = \mathbb{Z}[G]$ with basis as $\mathbb{Z}[G]$-Mod single generator denoted $[\,]$.

$B_n^u G = \mathbb{Z}[G]^{\otimes n+1}$ with basis as $\mathbb{Z}[G]$-Mod: generators $[g_1 | \cdots | g_n]$

differential $d = \sum_{i=0}^{n} (-1)^i d_i : B_n^u G \to B_{n-1}^u$

where d_i is defined on generators by:

- $d_0 [g_1 | \cdots | g_n] = g_1 [g_2 | \cdots | g_n]$
- $d_i [g_1 | \cdots | g_n] = [g_1 | \cdots | g_i g_{i+1} | \cdots | g_n]$ for $1 \leq i < n-1$
- $d_n [g_1 | \cdots | g_n] = [g_1 | \cdots | g_{n-1}]$

prop This is a free resolution of $\mathbb{Z}^{triv}$